Tutorial I

Problem 3:

If $\mathcal{T}_1, \mathcal{T}_2$ are two topology on X.

$$\left\{\begin{array}{l}\mathcal{T}_1 \text{ is a base for } \mathcal{T}_2 \\ \mathcal{T}_2 \text{ is a base for } \mathcal{T}_1\end{array}\right. \Rightarrow \mathcal{T}_1 = \mathcal{T}_2$$

- $d' \leq d$ so the d'-balls are larger than the d-balls and the topology induced by d is a base of the one induced by d'.

- Let $x \in X$, $r > 0$, $B_{d'}(x, \min(\alpha, r)) \subset B_d(x, r)$

$$\underset{\{y \in X \mid d'(x,y) \leq \min(\alpha, r)\}}{\underset{\parallel}{B_{d'}(x, \min(\alpha, r))}} \subset \underset{\{y \in X \mid d(x,y) \leq r\}}{\underset{\parallel}{B_d(x, r)}}$$

Then the topology induced by d' is a base of the one induced by d.

Tutorial II

Problem 4:

if $A \subset X$ st A, A^c both infinite then $\mathring{A} = \emptyset$, $\overline{A} = X$.

Indeed: $O \subset A$ open $\Rightarrow$ $A^c \subset O^c$ infinite so $O = \emptyset$

$A \subset F$ close $\Rightarrow$ F infinite close $\Rightarrow F = X$.

Tutorial III

Problem 3:

K compact $\iff \forall (O_i)_{i\in I}$ open, $\bigcup_{i\in I} = K \Rightarrow \exists J \subset I$ finite $\mid \bigcup_{i\in J} O_i = K$

$\iff \forall (F_i)_{i\in I}$ close, $\bigcap_{i\in I} F_i = \emptyset \Rightarrow \exists J \subset I$ finite $\mid \bigcap_{i\in J} F_i = \emptyset$

↑

set $F_i = O_i^c$

$\iff \forall (F_i)_{i\in I}$ close, $\forall J \subset I$ finite, $\bigcap_{i\in J} F_i \neq \emptyset \Rightarrow \bigcap_{i\in I} F_i \neq \emptyset$

↑

contraposition

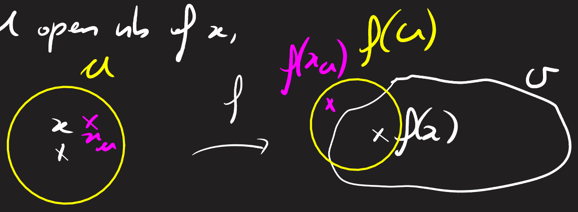
Tutorial IV

Problem 1:

1) $\boxed{\Rightarrow}$ Let V open nbh of $f(x)$ then $\exists\, \mathcal{U}$ open nb of x with $f(\mathcal{U}) \subset V$. Up to a rank $x_n \in \mathcal{U}$ so $f(x_n) \in V$.

2) $\boxed{\Leftarrow}$

Contraposition: assume $\exists V$ open nbh of $f(x)$ st $\forall\, \mathcal{U}$ open nb of x,
$\exists\, x_{\mathcal{U}} \in \mathcal{U} \mid f(x_{\mathcal{U}}) \notin V$
Let $(O_n)_n$ be a basis decreasing basis at x and
let $x_n := x_{O_n}$, then $x_n \to x$ but $f(x_n) \not\to f(x)$.



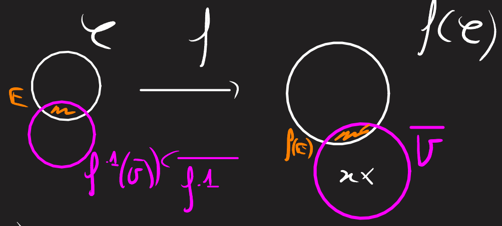
3) $x \neq y \Rightarrow f(x) \neq f(y)$, $\exists\, \mathcal{U}_x, \mathcal{U}_y$ separating $f(x), f(y)$ then $f^{-1}(\mathcal{U}_x)$, $f^{-1}(\mathcal{U}_y)$ separate x, y.

Problem 2:

1) We prove that f is proper: K compact $\Rightarrow f^{-1}(K)$ compact.
$f^{-1}(K)$ closed since f C^0 and K closed, and bounded otherwise $\exists (x_n)_n \subset \mathbb{R}^n$ st $\|x_n\| \to +\infty$ and $f(x_n) \in K$ which contradict the boundedness of K.
so $f^{-1}(K)$ is compact.

2) We prove that f is closed. Let $\mathcal{C} \subset \mathbb{R}^n$ closed. Prove that $\mathbb{R}^n \setminus f(\mathcal{C})$ open:
Let $x \in \mathbb{R}^n \setminus f(\mathcal{C})$, and V open nbh of x with compact closure

$$E := \underbrace{\mathcal{C}}_{\text{close}} \cap \underbrace{f^{-1}(\bar{V})}_{\text{compact}} \text{ is compact and } (f\ C^0)$$



$f(E) = f(\mathcal{C}) \cap \bar{V}$ is also compact. $\mathcal{U} := \underbrace{V}_{\text{open}} \setminus \underbrace{f(E)}_{\text{closed}}$ is an open nbh of x since $x \in V$ and $x \notin f(E) \subset f(\mathcal{C})$.

Moreover $\mathcal{U} \subset \bar{V} \setminus f(E) = \bar{V} \setminus f(\mathcal{C}) \subset \mathbb{R}^n \setminus f(\mathcal{C})$ so $\mathbb{R}^n \setminus f(\mathcal{C})$ is open.

3) f C^0, bijective, open map $\Rightarrow f^{-1}$ C^0 so f is a homeomorphism.

Problem 3:

see next tutorial.

Tutorial V

Problem 3:

1) K is uniformly continuous (continuous on a compact) so $\exists\, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ st $\eta(r) \xrightarrow[r\to 0]{} 0$ and $\forall x,y,z,t \in [0,1]$,

$|K(x,y) - K(z,t)| \leqslant \eta\left(\|(x,y)-(z,t)\|_{\mathbb{R}^2}\right)$. Let $x,y \in [0,1]$, then

$$|Tf(x) - Tf(y)| = \left|\int_0^1 (K(x,z) - K(y,z))f(z)\,dz\right| \leqslant \eta(|x-y|) \underbrace{\int_0^1 |f(z)|dz}_{<+\infty \text{ since is } C^0} \xrightarrow[|x-y|\to 0]{} 0$$

2) if $\|f\|_u \leqslant 1$ then from 1) we have

$$|Tf(x) - Tf(y)| \leqslant \eta(|x-y|) \xrightarrow[|x-y|\to 0]{} 0 \quad \text{uniformly in } f \text{ (equicontinuity)}$$

With Ascoli theorem $\{Tf, f \in C([0,1],\mathbb{R}) \mid \|f\|_u \leqslant 1\}$ is relatively compact.

3) Let $K = \overline{B}_u(0,1)$ non empty closed convex subset of $CB([0,1],\mathbb{R})$ (Banach)

Let $f \in K$, $\|Tf\|_u \leqslant \|K\|_u \|f\|_u \leqslant 1$ so $Tf \in K$ and by 2) $T(K) \subset K$ is relatively compact so by Schauder fixed point theorem, T has a fixed point.

Problem 4:

(f_n) cauchy in $C(X,Y)$, then $\forall x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ Cauchy so

$f_n(x) \xrightarrow[n\in\mathbb{N}]{} f(x)$, then

$$\|f_n(x) - f(x)\| \leqslant \sup_{m\geqslant n} \|f_n(x) - f_m(x)\| \leqslant \sup_{m\geqslant n} \|f_n - f_m\| \xrightarrow[n\to+\infty]{} 0$$

so $\|f_n - f\| \xrightarrow[n\to+\infty]{} 0$.

and f continuous since $\|f(x) - f(y)\| \leqslant \|f_n(x) - f_n(y)\| + \underbrace{2\|f_n - f\|}_{\text{uniform in } x\to y}$

Tutorial VI

Problem 1:

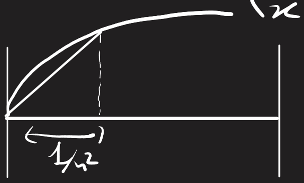
Let $T, U \in \mathcal{L}(X, Y)$, $x, y \in X$.

$$\|Tx - Uy\|_Y \leqslant \|T(x-y)\|_Y + \|(T-U)y\|_Y$$

$$\leqslant \|T\|_{\mathcal{L}} \|x-y\|_X + \|T-U\|_{\mathcal{L}} \|y\|_Y \xrightarrow[(U,y)\to(T,x)]{} 0$$

Problem 2:

1)



$f_n(x) = \min(nx, \sqrt{x})$, $nx \leqslant \sqrt{x} \Rightarrow x \leqslant \frac{1}{n^2} \Rightarrow |\sqrt{x} - nx| \leqslant \frac{2}{n}$

so $f_n \overset{u}{\rightarrow} \sqrt{\cdot}$.

2) (f_n) cauchy, 1 lipschitz, $f_n \overset{u}{\rightarrow} f$, $\|f(x) - f(y)\| \leqslant 2\|f_n - f\| + \|f_n(x) - f_n(y)\|$

$$\leqslant 2\|f_n - f\| + \|x-y\| \xrightarrow[n\to+\infty]{} \|x-y\|$$

4) Let $(f_n)_n \subset (L, N)$ Cauchy then $(f_n)_n$ is cauchy in $(C^0, \|.\|_\infty)$

so $f_n \overset{u}{\rightarrow} f \in C^0$. Let $x \neq y$

$$|f(x) - f_n(x) - (f(y) - f_n(y))| \leqslant \sup_{m \geqslant n} |f_m(x) - f_n(x) - (f_m(y) - f_n(y))|$$

$$\leqslant |x-y| \sup_{m \geqslant n} Lip(f_m - f_n)$$

so $Lip(f - f_n) \leqslant \sup_{m \geqslant n} Lip(f_m - f_n) \xrightarrow[n\to+\infty]{} 0$

5) Let $(f_n)_n$ be Cauchy for $\|.\|$, then $f_n \xrightarrow[n\to+\infty]{u} f$, $f_n' \xrightarrow[n\to+\infty]{u} g$ where $f, g \in C^0$.

we just need $f' = g$, this follows from:

$$\left| f(x) - \int_0^x g(y)dy - f(0) \right| \leqslant \|f - f_n\|_\infty + \left| f_n(x) - \int_0^x g(y)dy - f(0) \right|$$

$$= \|f - f_n\|_\infty + \left| \int_0^x (f_n' - g) + f_n(0) - f(0) \right| \leqslant 2\|f - f_n\|_\infty + x\|f_n' - g\|_\infty$$

$$\leqslant \|f - f_n\|_\infty + \|f_n' - g\|_\infty \xrightarrow[n\to+\infty]{} 0$$ so $f(x) = f(0) + \int_0^x g(y)dy$ and $f' = g$.

Problem 3:

1) $f = 0$

2) $f =$

3). take a sequence in $\mathbb{R}^{(\mathbb{N})}$ that converges uniformly to $\left(\frac{1}{n+1}\right)_{n \in \mathbb{N}}$.

. define $Te_n = \frac{e_n}{n+1}$ where $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of $\mathbb{R}^{\mathbb{N}}$.

Tutorial VII

Problem 1: $\|x\| \geqslant \sup_{\substack{\varphi \in F' \\ \|\varphi\| \leqslant 1}} \varphi(x)$, si $x=0$ $\varphi=0$ convient, sinon

on pose $f_x : \begin{array}{l} \mathbb{R}x \to \mathbb{R} \\ tx \to t\|x\| \end{array}$, $f_x \in (\mathbb{R}x)'$, $\|f_x\|_{(\mathbb{R}x)'} = 1$

f_x linéaire continue donc par Hahn Banach, g se prolonge sur F en $\tilde{f_x} : F \to \mathbb{R}$ tq $\|\tilde{f}\|_{F'} = 1$

de plus $\tilde{f_x} x = f_x x = \|x\|$ donc $\|x\| \leqslant \sup_{\varphi \in F, \|\varphi\| \leqslant 1} \varphi x$

2) $\exists\, y \in X^{**}$ st $|y(\varphi)| = \|\varphi\|$, $x := i^{-1}(y) \in X$

$i : X \to X^{**}$

$x \mapsto \begin{array}{l} X^* \to \mathbb{R} \\ u \mapsto u(x) \end{array}$

then $y(\varphi) = \varphi(x)$.

Problem 2: $\varphi : \{\text{convergent sequences}\} \subset \ell^\infty(\mathbb{R}) \to \mathbb{R}$ extended on $\ell^\infty(\mathbb{R})$ by H.B.

$(u_n)_{n\in\mathbb{N}} \mapsto \lim_{n\in\mathbb{N}} u_n$

If $\varphi : \ell^\infty(\mathbb{R}) \to \mathbb{R}$ with $(u_n)_n \in \ell^1(\mathbb{N})$. then $\varphi(\underline{e_n}) = 0 = u_n\ \forall n \in \mathbb{N}$

$(v_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} u_n v_n$

$e_i = (0, \dots, 0, \underset{i^{th}\text{ position}}{1}, 0 \dots)$

so $\varphi = 0$ absurd.

Problem 3: 1) let $f \in C([0,1], \mathbb{R})$, $f = \underbrace{(f - f(0))}_{\in X} + \underbrace{f(0)}_{\in \mathcal{C}}$ and $\mathcal{C} \cap X = \{0\}$.

2) $F : C([0,1], \mathbb{R}) \to \mathbb{R}$.

$f \mapsto f(0)$

Problem 4: $\varphi : X^* \to \mathcal{U}^*$ injective, surjective by Hahn-Banach.

$f \mapsto f|_{\mathcal{U}}$

$\|f\|_{X^*} = \|\varphi(f)\|_{\mathcal{U}^*}$ by density.

Tutorial VIII-IX

Problem 1:

$$\forall x \in E, \quad \|Tx\|_F = \sup_{\substack{\varphi \in F^* \\ \|\varphi\|_{F^*} \leq 1}} \varphi(Tx) < +\infty$$

$\{\varphi \circ T, \ \varphi \in F^* \mid \|\varphi\|_{F^*} \leq 1\} \subset E^*$, by the uniform boundedness:

$$\sup_{\substack{\varphi \in F^* \\ \|\varphi\|_{F^*} \leq 1}} \|\varphi \circ T\|_E < +\infty \text{ and } \|Tx\|_F = \sup_{\substack{\varphi \in F^* \\ \|\varphi\|_{F^*} \leq 1}} \varphi(Tx) \leq \left(\sup_{\substack{\varphi \in F^* \\ \|\varphi\|_{F^*} \leq 1}} \|\varphi \circ T\|_E \right) \|x\|_E$$

Problem 2:

. 1) ⇒ 2) . Let $Tx_n \to y \in F$, $(Tx_n)_n$ thus $(x_n)_n$ is Cauchy so $x_n \to x \in E$. Since T is continuous $Tx = y \in Im(T)$.

. 2) ⇒ 1) $T : E \to \underbrace{Im(T)}_{\text{closed and Banach}}$ open and bijective thus $T^{-1} \in L_c(F,E)$

$$\|x\|_E = \|T^{-1}Tx\|_E \leq \|T^{-1}\|_{L_c(F,E)} \|Tx\|_F$$

Problem 3:

1) linearity of the limit

2) $\forall x \in E \quad \sup_{n \in \mathbb{N}} \|T_n x\|_F < +\infty$ by U.B. $\quad \sup_{n \in \mathbb{N}} \|T_n\|_{L_c(E,F)} < +\infty$

and $\|Tx\|_E \leq \lim_{n \to +\infty} \|T_n x\|_F \leq \underbrace{\lim_{n \to +\infty} \|T_n\|_{L_c(E,F)}} \|x\|_E$

$$\leq \sup_{n \in \mathbb{N}} \|T_n\|_{L_c(E,F)} < +\infty$$

4) $T_N((b_n)_n) = \sum_{n=0}^{N} \overline{a_n} b_n \quad$ (continuous as map on $\mathbb{R}^{N+1} \to \mathbb{R}$)

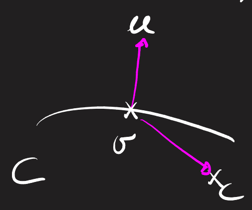
5) $b_n := |a_n|^{q-1} s(a_n) \mathbb{1}_{n \leq N} \quad \left(\overline{s(a_n)}\, a_n = |a_n| \right)$, $(b_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{C})$ so

$$T((b_n)) = \sum_{n=0}^{N} |a_n|^q \leq \|T\| \left(\sum_{n=0}^{N} |b_n|^p \right)^{1/p} = \|T\| \left(\sum_{n=0}^{N} |a_n|^{\overbrace{(q-1)p}^{q}} \right)^{1/p}$$

so $\left(\sum_{n=0}^{N} |a_n|^q \right)^{1/q} \leq \|T\| \quad$ take $N \to +\infty$.

Tutorial X

Problem 1:



1) $\boxed{\Rightarrow}$ $\|u-v\|^2 \leqslant \|u-(tc+(1-t)v)\|^2 = \|u-v+t(v-c)\|^2$

so $2\cancel{t}\langle c-v|u-v\rangle \leqslant t^{\cancel{2}}\|v-c\|^2 \quad \forall t\in[0,1]$

so $\langle c-v|u-v\rangle \leqslant 0$

$\langle v-u|p_c(v)-p_c(u)\rangle \geqslant \|p_c(v)-p_c(u)\|^2$

$\boxed{\Leftarrow}$ $\|u-c\|^2 = \|u-v\|^2+\|c-v\|^2 \underbrace{-2\langle u-v|c-v\rangle}_{\geqslant 0} \geqslant \|u-v\|^2$

3) $\nearrow$

2) $\begin{cases} \langle u-p_c(u), p_c(v)-p_c(u)\rangle \leqslant 0 \\ \langle v-p_c(v), p_c(u)-p_c(v)\rangle \leqslant 0 \end{cases} \Rightarrow \overbrace{\langle u-v+p_c(v)-p_c(u)|p_c(v)-p_c(u)\rangle}\leqslant 0$

$\Rightarrow \|p_c(v)-p_c(u)\|^2 \leqslant \langle v-u|p_c(v)-p_c(u)\rangle \leqslant \|v-u\|\,\|p_c(v)-p_c(u)\|$

$\Rightarrow \|p_c(v)-p_c(u)\| \leqslant \|v-u\|.$

Problem 2:

1) see https://math.stackexchange.com/questions/324538/separable-hilbert-space-have-a-countable-orthonormal-basis

2) Let $(E,\|.\|)$ be a Banach space. Assume by contradiction that $(e_n)_{n\in\mathbb{N}^*}$ is an algebraic basis of E.

Soit $x_n := \sum_{k=1}^{n}\frac{e_k}{k^2} \in E$, $(x_n)_{n\in\mathbb{N}}$ is Cauchy:

$$\|x_{n+p}-x_n\| = \left\|\sum_{k=n+1}^{n+p}\frac{e_k}{k^2}\right\| \leqslant \sum_{k=n+1}^{n+p}\frac{1}{k^2} \leqslant \sum_{k\geqslant n}\frac{1}{k^2} \xrightarrow[n\to+\infty]{} 0 \quad \text{since } \sum_{k\in\mathbb{N}^*}\frac{1}{k^2} < +\infty$$

since E is Banach, $\exists\, x\in E$ s.t. $\|x_n - x\|_E \xrightarrow[n\to+\infty]{} 0$.

Using the algebraic basis $(e_n)_{n\in\mathbb{N}^*}$, $\exists\, k_1,\dots,k_r \in \mathbb{N}$, $\exists\, \lambda_1,\dots,\lambda_r \in \mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$) s.t. $x = \sum_{i=1}^{r}\lambda_i e_{k_i}$.

Lemma: let E be a Banach space and F, G closed subvector subspaces such that $\underbrace{E = F \oplus G}$.

this means: $\bullet\ \forall x \in E \ \exists! \, (x_F, x_G) \in F \times G$ such that $x = x_F + x_G$

$\bullet\ F \cap G = \{0_E\}$

Then the projections $\Pi_F : \begin{array}{l} F \oplus G \to F \\ x \mapsto x_F \end{array}$, $\Pi_G : \begin{array}{l} F \oplus G \to G \\ x \mapsto x_G \end{array}$ are continuous.

proof: F, G are Banach spaces with the induced norm.

$F \times G$ also with $\|(x,y)\|_{F \times G} := \|x\|_E + \|y\|_E$ for $(x,y) \in F \times G$.

Consider $\varphi : \begin{array}{l} G \times K \to G \oplus K \\ (x,y) \mapsto x+y \end{array}$ linear and bijective.

and continuous: $\|\varphi(x,y)\|_E = \|x+y\|_E \leqslant \|x\|_E + \|y\|_E = \|(x+y)\|_{F \times G}$

By the open mapping theorem φ^{-1} is continuous.

$p_F : \begin{array}{l} F \times G \to F \\ (x,y) \mapsto x \end{array}$ is continuous: $\|p_F(x,y)\|_E = \|x\|_E \leqslant \|x\|_E + \|y\|_E = \|(x,y)\|_{F \times G}$

so $\Pi_F = p_F \circ \varphi^{-1}$ is continuous by composition.

Let $k > \max(k_1, \dots, k_r)$ consider Π_k the projection on $\underbrace{\mathrm{span}(e_k)}_{\text{closed because finite dimensional}}$.

then $\Pi_k(x_n - x) = \Pi_k x_n - \underbrace{\Pi_k x}_{=0} = \dfrac{e_k}{k^2} \underset{n \to +\infty}{\not\to} 0$ this contradicts C^0 of Π_k.

Tutorial XI

Problem 1:

1) let $u \in \ell^1$, $\|Su\|_{\ell^2} = \|u\|_{\ell^2} \leqslant \|u\|_{\ell^1}$

so $\|S\| \leqslant 1$ since $\|Se_0\|_{\ell^2} = 1 = \|e_0\|_{\ell^1}$ we have $\|S\| = 1$

2) $S^* : (\ell^2)^* \to (\ell^1)^*$ $\quad (S^*u)v \quad$ If $u \in \ell^2$, $v \in \ell^\infty$

$v \mapsto vS$

$S^* = \phi_1 T \phi_2$ $\quad$ with $T : \ell^2 \to \ell^\infty$, $\phi_1 : \ell^\infty \to (\ell^1)^*$, $\phi_2 : (\ell^2)^* \to \ell^2$

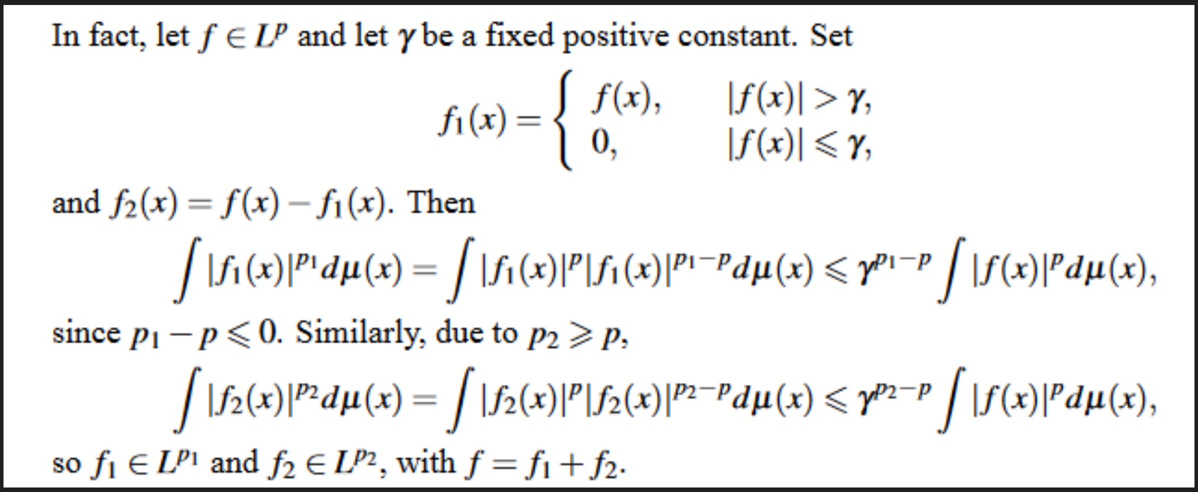
Let $u \in \ell^2$, $Tu = \phi_1^{-1} S^*(\phi_2^{-1} u) = \phi_1^{-1}(\phi_2^{-1} u) S$

$(Tu)_n = (\phi_2^{-1} u) S e_n = (\phi_2^{-1} u) e_{n+1} = \langle u, e_{n+1} \rangle = u_{n+1}$

so $Tu = (u_1, u_2, \dots)$.

$\|T\| = +\infty$ $\quad$ take $\left(\frac{1}{n}\right)_{n \in \mathbb{N}^*} \in \ell^2 \setminus \ell^1$

Problem 2:



In fact, let $f \in L^p$ and let γ be a fixed positive constant. Set

$$f_1(x) = \begin{cases} f(x), & |f(x)| > \gamma, \\ 0, & |f(x)| \leqslant \gamma, \end{cases}$$

and $f_2(x) = f(x) - f_1(x)$. Then

$$\int |f_1(x)|^{p_1} d\mu(x) = \int |f_1(x)|^p |f_1(x)|^{p_1 - p} d\mu(x) \leqslant \gamma^{p_1 - p} \int |f(x)|^p d\mu(x),$$

since $p_1 - p \leqslant 0$. Similarly, due to $p_2 \geqslant p$,

$$\int |f_2(x)|^{p_2} d\mu(x) = \int |f_2(x)|^p |f_2(x)|^{p_2 - p} d\mu(x) \leqslant \gamma^{p_2 - p} \int |f(x)|^p d\mu(x),$$

so $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$, with $f = f_1 + f_2$.

Tutorial XII

Problem 1:

1) $i(x_n): E' \to \mathbb{R}$

$u \mapsto u(x_n)$

Let $u \in E'$

$i(x_n)u = u(x_n) \to u(x) \in \mathbb{R}$

so $\sup\limits_{n\in\mathbb{N}} i(x_n)u < +\infty$

by U.B. $\sup\limits_{n\in\mathbb{N}} \|i(x_n)\|_{E''} = \sup\limits_{n\in\mathbb{N}} \|x_n\|_E < +\infty$

2) contradiction $\|x_n\| \to +\infty$.

3) Let $y \in X^*$, Let $\varepsilon > 0$, $\exists \underset{m}{m} \in \mathbb{N} \mid \forall k \geq m, |\langle y, x_k - \underset{n}{\ } x\rangle| \leq \varepsilon$

$$y\left(\frac{1}{n}\sum_{k=1}^{n} x_k - x\right) = \underbrace{\underbrace{\frac{1}{n}\sum_{k=1}^{m}\langle y, x_k - x\rangle}_{<+\infty}}_{\xrightarrow[n\to+\infty]{}0} + \frac{1}{n}\sum_{k=m+1}^{n}\underbrace{\langle y, x_k - x\rangle}_{|\cdot|\leq\varepsilon}$$

Problem 2:

$\mathbb{1}_{[n,n+1]} \rightharpoonup f$ (bounded in L^2), $\forall \varphi \in C_c^\infty(\mathbb{R}) \langle \varphi, f\rangle = 0$

so $f = 0$.

4) Let $f \in L^2$, $f = \sum \langle e_k, f\rangle e_k$. $\|f\|^2 = \sum_k |\langle e_k, f\rangle|^2$

so $\langle e_k, f\rangle \xrightarrow[k\to+\infty]{} 0 \Rightarrow \cos(2\pi k x) \rightharpoonup 0$

Tutorial XIII

2) $\mathcal{E}(\rho) \leqslant \|\rho\|_2^2 + \|V\|_2 \|\rho\|_2 + \|\rho\|_2 \underbrace{\|w*\rho\|_2}_{\leqslant \|w\|_1 \|\rho\|_2}$

4) $\mathcal{E}(\rho) \geqslant \|\rho\|_2^2 - \|V\|_2 \|\rho\|_2 - \|w\|_1 \|\rho\|_2^2$

$$\geqslant \|\rho\|^2 \left(1 - \frac{\varepsilon}{2} - \|w\|_1\right) - \frac{\|V\|_2^2}{2\varepsilon} = \|\rho\|^2 \frac{\varepsilon}{2} - \frac{\|V\|_2^2}{2\varepsilon}$$

$ab \leqslant \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ chose $\varepsilon = 1 - \|w\|_1$

$$\Rightarrow \|\rho\|_2^2 \leqslant \frac{2}{\varepsilon}\left(\mathcal{E}(\rho) + \frac{\|V\|_2^2}{2\varepsilon}\right) = \frac{1}{1-\|w\|_1}\left(2\mathcal{E}(\rho) + \frac{\|V\|_2^2}{1-\|w\|_1}\right)$$

7) $\int (\mu-\rho_n) * w (\mu-\rho_n) = \int (\mu-\rho_n) * w \mu + \int \rho_n * w \rho_n - \int \mu * w \rho_n \geqslant 0$

so $\int \mu * w \mu \leqslant 2 \underbrace{\int \underbrace{\mu * w}_{\in L^2} (\mu - \rho_n)}_{\xrightarrow[n\to+\infty]{} 0} + \int \rho_n * w \rho_n$ take $\underline{\lim}$.

8) $\int V\rho_n \leqslant \mathrm{E}(\rho_n) + (\|w\|_1 - 1)\|\rho_n\|^2 \leqslant \mathrm{E}(\rho_n)$

Let $r > 0$, $\inf\limits_{|x|\geqslant r} V(x) \int \mathbb{1}_{|x|>r} \rho(x)\,dx + \int \mathbb{1}_{|x|\leqslant r} V(x) \rho(x)\,dx \leqslant \mathrm{E}(\rho)$

$$\Rightarrow \int \mathbb{1}_{|x|>r} \rho(x)\,dx \leqslant \frac{2\,\mathrm{E}(\rho)}{\inf\limits_{|x|>r} V(x)}$$

9)

$$\underbrace{1 - \int_{\mathbb{R}} \mathbb{1}_{|x|>r} \rho_n(x)\,dx}_{\xrightarrow[r\to+\infty]{} 0 \text{ uniformly in } n} = \int_{\mathbb{R}} \mathbb{1}_{|x|\leqslant r} \rho_n(x)\,dx \xrightarrow[n\to+\infty]{} \int \mathbb{1}_{|x|\leqslant r} \rho(x)\,dx$$

so $\int \rho = 1$

Homework

Problem 5:

2) Define $\forall x, y \in H$,

$$\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)$$

- Let $x, y \in H$,

$$\langle y,x\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|y+ix\|^2 + i\|y-ix\|^2\right)$$
$$= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|-iy+x\|^2 + i\|iy+x\|^2\right)$$
$$= \overline{\langle x,y\rangle}$$

- Let $x \in H$,

$$\langle x,x\rangle = \frac{1}{4}\left(\|2x\|^2 - i\|(1+i)x\|^2 + i\|(1-i)x\|^2\right)$$
$$= \frac{1}{4}\left(4\|x\|^2 - 2i\|x\|^2 + 2i\|x\|^2\right) = \|x\|^2$$

so $\langle . \rangle$ is skew-symmetric positive definite.

- For the linearity we start with the real part:

Define, $\forall x, y \in H$, $\langle x,y\rangle_R := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$.

Let $x, y, z \in H$.

$$\langle x, y+z\rangle_R = \langle x,y\rangle_R + \langle x,z\rangle_R$$
$$\Leftrightarrow \|x+y+z\|^2 - \|y+z-x\|^2 = \|x+y\|^2 - \|y-x\|^2 + \|x+z\|^2 - \|z-x\|^2$$
$$\Leftrightarrow \cancel{\|2y+z\|^2} + \|2x+z\|^2 = \cancel{\|2y+z\|^2} + \|z-2x\|^2 + 2\left(\|x+z\|^2 - \|z-x\|^2\right)$$

{Parallelogram identities:

$$\begin{cases} \|x+y+z\|^2 + \|y-x\|^2 = \frac{1}{2}\left(\|2y+z\|^2 + \|2x+z\|^2\right) \\ \|y+z-x\|^2 + \|x+y\|^2 = \frac{1}{2}\left(\|2y+z\|^2 + \|z-2x\|^2\right) \end{cases}$$

$$\Leftrightarrow 2\|\cancel{x+z}\|^2 + 2\cancel{\|x\|}^2 - \|\cancel{z}\|^2 = 2\cancel{\|x\|}^2 + 2\|\cancel{x-z}\|^2 - \|\cancel{z}\|^2 + 2\|\cancel{x+z}\|^2 - 2\|\cancel{z-x}\|^2$$

$$\begin{cases} \|2x+z\|^2 + \|z\|^2 = 2\|x+z\|^2 + 2\|x\|^2 \\ \|2x-z\|^2 + \|z\|^2 = 2\|x\|^2 + 2\|x-z\|^2 \end{cases}$$

This summability implies that $\forall \lambda \in \mathbb{Q}$, $\langle x, \lambda y\rangle_R = \lambda\langle x,y\rangle_R$

$\mathbb{Q}$ is dense in $\mathbb{R}$ so we may conclude using the continuity of $\langle x, \cdot\rangle_R$ (which is a consequence of Cauchy-Schwarz inequality since any symmetric, positive-definite, $\mathbb{Q}$-bilinear form satisfies Cauchy-Schwarz) that $\forall \lambda \in \mathbb{R}$, $\langle x, \lambda y\rangle_R = \lambda\langle x,y\rangle_R$.

Then, we notice that $\langle x, y\rangle = \langle x,y\rangle_{\mathbb{R}} + i\langle x, -iy\rangle_{\mathbb{R}}$ thus $\langle . \rangle$ is $\mathbb{R}$- linear in its 2^{nd} variable. We extend to $\mathbb{C}$ linearity noticing that

$$4\langle x, iy\rangle = \|x+iy\|^2 - \|x-iy\|^2 - i\|x-y\|^2 + i\|x+y\|^2 = i\langle x,y\rangle.$$

Problem 6:

1) The family $\left(\mathbb{1}_{[a,x]}\right)_{x\in[a,b]}$ is countable and $\forall x,y \in [a,b]$,

$$x\neq y \Rightarrow \|\mathbb{1}_{[a,x]} - \mathbb{1}_{[a,y]}\|_{L^\infty} = 1.$$

2) $|\phi(x)y| = \left|\sum_{n\in\mathbb{N}} x_n y_n\right| \underset{\substack{\uparrow \\ \text{Hölder}}}{\leq} \|x\|_{\ell^\infty}\|y\|_{\ell^1}$ so $\phi(x)\in \ell^1(\mathbb{R})^*$ and

$$\|\phi(x)\|_{\ell^1(\mathbb{R})^*} \leq \|x\|_{\ell^\infty}.$$

3) $\forall n\in\mathbb{N}$, $|\phi(x)\underset{\substack{\| \\ (0,\,-\,,0,\,\underset{\substack{\uparrow \\ n^{th}\text{ position}}}{1},\,\dots)}}{e_n}| = |x_n| = |x_n|\cdot\overbrace{\|e_n\|_{\ell^1(\mathbb{R})}}^{=1}$ so $\|\phi(x)\|_{\ell^1(\mathbb{R})^*} \geq |x_n|$

thus $\|\phi(x)\|_{\ell^1(\mathbb{R})^*} \geq \sup_{n\in\mathbb{N}} |x_n| = \|x\|_{\ell^\infty}$ and $\|\phi(x)\|_{\ell^1(\mathbb{R})^*} = \|x\|_{\ell^\infty(\mathbb{R})}$.