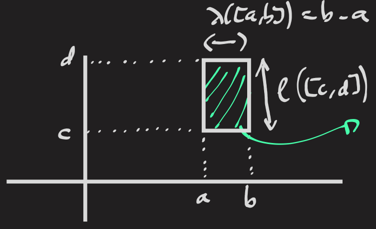
VIII / Product measures



$\ell_{\mathbb{R}^2}([a,b]\times[c,d]) := \ell([a,b])\cdot\ell([c,d]) = (b-a)(d-c)$

Def product σ-Algebra

Let $(E,\mathcal{A})$, $(F,\mathcal{B})$ be two measurable spaces

$\mathcal{A}\otimes\mathcal{B} := \sigma(\{M\times N \mid (M,N)\in\mathcal{A}\times\mathcal{B}\}) \subset \mathcal{P}(E\times F)$ (measurable rectangle: $M\times N$)

Remark: • $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$

$\supseteq$ $\mathcal{B}(\mathbb{R}^2) = \sigma(\{]a,b[\times]c,d[\mid a<b,\ c<d\}) \subset \sigma(\{B_1\times B_2 \mid B_1,B_2\in\mathcal{B}(\mathbb{R})\}) = \mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$

$\subseteq$ $\quad \Gamma := \{M\subset\mathbb{R} \mid M\times\mathbb{R}\in\mathcal{B}(\mathbb{R}^2)\}$ is a σ-Algebra:

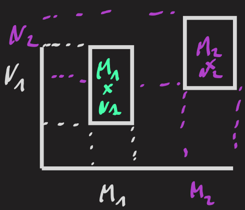
- $\mathbb{R}\in\Gamma$
- $M\in\Gamma \Rightarrow M^c\times\mathbb{R} = \underbrace{(M\times\mathbb{R})}_{\in\mathcal{B}(\mathbb{R}^2)}{}^c \in \mathcal{B}(\mathbb{R}^2) \Rightarrow M^c\in\Gamma$
- $(M_n)_{n\in\mathbb{N}}\in\Gamma \Rightarrow \left(\bigcup_{n\in\mathbb{N}} M_n\right)\times\mathbb{R} = \bigcup_{n\in\mathbb{N}} \underbrace{(M_n\times\mathbb{R})}_{\in\mathcal{B}(\mathbb{R}^2)} \in\mathcal{B}(\mathbb{R}^2) \Rightarrow \bigcup_{n\in\mathbb{N}} M_n \in \mathbb{R}$

Γ contains all open sets so $\mathcal{B}(\mathbb{R})\subset\Gamma$ meaning $\forall M\in\mathcal{B}(\mathbb{R}),\ M\times\mathbb{R}\in\mathcal{B}(\mathbb{R}^2)$

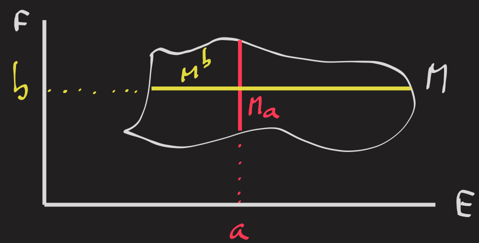
considering $\{M\subset\mathbb{R} \mid \mathbb{R}\times M\in\mathcal{B}(\mathbb{R}^2)\}$ instead of Γ, $\forall N\in\mathcal{B}(\mathbb{R}),\ \mathbb{R}\times N\in\mathcal{B}(\mathbb{R}^2)$

so $\forall M,N\in\mathcal{B}(\mathbb{R}),\ M\times N = (M\times\mathbb{R})\cap(\mathbb{R}\times N)\in\mathcal{B}(\mathbb{R}^2)$

• $\mathcal{A}\otimes\mathcal{B}$ contains is more than rectangles:



$(M_1\times N_1)\cup(M_2\times N_2)$ is not necessarily a rectangle



Proposition: sections

Let $(a,b)\in E\times F$,

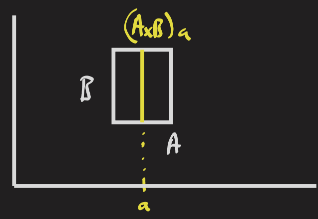
1) if $M\in\mathcal{A}\otimes\mathcal{B}$, then $M_a := \{y\in F \mid (a,y)\in M\}\in\mathcal{B}$, $\quad M^b := \{x\in E \mid (x,b)\in M\}\in\mathcal{A}$

2) if $f:(E\times F,\mathcal{A}\otimes\mathcal{B})\to(G,\Gamma)$ is measurable, then $f_a: y\mapsto f(a,y)$, $f^b: x\mapsto f(x,b)$ are measurable

proof: 1) $\{M\in\mathcal{A}\otimes\mathcal{B} \mid M_a\in\mathcal{B}\}$ is a σ-Algebra:

- $(E\times F)_a = F\in\mathcal{B}$
- Let $M\in\mathcal{A}\otimes\mathcal{B}$, $(M^c)_a\in\mathcal{B} \Rightarrow (M_a)^c = (M^c)_a\in\mathcal{B}$
- Let $(M_n)_{n\in\mathbb{N}}\in\mathcal{A}\otimes\mathcal{B}$, $\forall n\in\mathbb{N}\ (M_n)_a\in\mathcal{B} \Rightarrow \left(\bigcup_{n\in\mathbb{N}} M_n\right)_a = \bigcup_{n\in\mathbb{N}}(M_n)_a\in\mathcal{B}$

containing $\{A\times B \mid (A,B)\in\mathcal{A}\times\mathcal{B}\}$: $\quad (A\times B)_a = \begin{cases} B & \text{if } a\in A \\ \emptyset & \text{if } a\notin A\end{cases} \in\mathcal{B}$



so $\{M\in\mathcal{A}\otimes\mathcal{B} \mid M_a\in\mathcal{B}\} = \mathcal{A}\otimes\mathcal{B}$, similarly for $M^b\in\mathcal{A}$.

2) Let $M\in\Gamma$, $f_a^{-1}(\Gamma) = \{y\in F \mid f(a,y)\in\Gamma\}$

$\qquad = \{y\in F \mid (a,y)\in f^{-1}(\Gamma)\} = (f^{-1}(\Gamma))_a$

def/theorem: product measure

Let $(E,\mathcal{A},\mu)$, $(F,\mathcal{B},\nu)$ be two σ-finite measured spaces

there exists a unique measure on $(E\times F, \mathcal{A}\otimes\mathcal{B})$ denoted $\mu\otimes\nu$ such that

$$\forall (A,B)\in \mathcal{A}\times\mathcal{B},\ (\mu\otimes\nu)(A\times B) = \mu(A)\nu(B)$$

Moreover $\forall M\in \mathcal{A}\otimes\mathcal{B}$, $(\mu\otimes\nu)(M) = \int_E \nu(M_x)\,d\mu(x) = \int_F \mu(M^y)\,d\nu(y)$

proof: unicity: assume m_1, m_2 are measures on $(E\times F, \mathcal{A}\otimes\mathcal{B})$ st $\forall (A,B)\in\mathcal{A}\times\mathcal{B}$, $m_1(A\times B) = m_2(A\times B) = \mu(A)\nu(B)$

. μ and ν are σ-finite so $\exists (E_n)_{n\in\mathbb{N}}\subset\mathcal{A}$, $(F_n)_{n\in\mathbb{N}}\subset\mathcal{B}$ such that $E = \bigcup_{n\in\mathbb{N}} E_n$, $F = \bigcup_{n\in\mathbb{N}} F_n$

$(E_n)_{n\in\mathbb{N}}\nearrow$, $(F_n)_{n\in\mathbb{N}}\nearrow$ and $\forall n\in\mathbb{N}$ $\mu(E_n)<+\infty$, $\nu(F_n)<+\infty$.

Then $E\times F = \bigcup_{n\in\mathbb{N}} (E_n\times F_n)$ then $m_1(E_n\times F_n) = m_2(E_n\times F_n) = \mu(E_n)\nu(F_n) < +\infty$

($E_n\times F_n$ $\nearrow$)

By Dynkin's theorem $m_1 = m_2$ on $\sigma(\{A\times B \mid A,B\in\mathcal{A}\times\mathcal{B}, A\subset E_n, B\subset F_n\}) = \mathcal{A}|_{E_n}\otimes\mathcal{B}|_{F_n} \overset{⊛}{=} (\mathcal{A}\otimes\mathcal{B})|_{E_n\times F_n}$

$= \{M\cap(E_n\times F_n), M\in\mathcal{A}\otimes\mathcal{B}\}$

⊛

$\subseteq$ $(\mathcal{A}\otimes\mathcal{B})|_{E_n\times F_n}$ is a σ-algebra and contains

$\supseteq$ $\{M\in\mathcal{A}\otimes\mathcal{B} \mid M\cap(E_n\times F_n)\in\mathcal{A}|_{E_n}\otimes\mathcal{B}|_{F_n}\}$ is a σ-algebra containing rectangles thus $= \mathcal{A}\otimes\mathcal{B}$

Let $M\in\mathcal{A}\otimes\mathcal{B}$, $m_1(M) = \lim_{n\to+\infty} m_1(M\cap(E_n\times F_n)) = \lim_{n\to+\infty} m_2(M\cap(E_n\times F_n)) = m_2(M)$

($M\cap(E_n\times F_n)\in\mathcal{A}\otimes\mathcal{B}|_{E_n\times F_n}$)

$x\mapsto \nu(M_x)$, $y\mapsto\mu(M^y)$ measurable . if ν is finite, $\mathcal{D} := \{M\in\mathcal{A}\otimes\mathcal{B} \mid x\mapsto\nu(M_x) \text{ measurable}\}$ is a Dynkin's system

- $x\mapsto\nu((E\times F)_x) = \nu(F)$ measurable so $E\times F\in\mathcal{D}$
- if $M,N\in\mathcal{D}, M\subset N$, $x\mapsto\nu((M\setminus N)_x) = \nu(M_x\setminus N_x) = \nu(M_x)-\nu(N_x)$ measurable so $M\setminus N\in\mathcal{D}$
- if $(M_n)_{n\in\mathbb{N}}\subset\mathcal{D}$, $\nearrow$, $x\mapsto\nu((\bigcup_{n\in\mathbb{N}} M_n)_x) = \nu(\bigcup_{n\in\mathbb{N}}(M_n)_x) = \lim_{n\to+\infty}\nu((M_n)_x)$ measurable so $\bigcup_{n\in\mathbb{N}} M_n\in\mathcal{D}$

$\{A\times B \mid (A,B)\in\mathcal{A}\times\mathcal{B}\}\subset\mathcal{D}$ so by Dynkin's theorem, $\mathcal{A}\otimes\mathcal{B} = \sigma(\{A\times B \mid (A,B)\in\mathcal{A}\times\mathcal{B}\})\subset\mathcal{D} = \mathcal{A}\otimes\mathcal{B}$

(contains $E\times F$, stable under $\cap$)

. if ν is σ-finite, $\nu(E_x) = \lim_{n\to+\infty} \nu(E_x\cap E_n)$ ($=: \nu_n(E_x)$ finite measure)

Existence: let $m(M) := \int_E \nu(M_x)\,d\mu(x)$. First, $m(A\times B) = \int_E \nu(B)\mathbb{1}_A(x)\,d\mu(x) = \int_A \nu(B)\,d\mu = \mu(A)\nu(B)$ ($A\in\mathcal{A}$, $B\in\mathcal{B}$)

$(A\times B)_x = B$ if $x\in A$, $=\emptyset$ if $x\notin A$ so $\nu((A\times B)_x) = \nu(B)\mathbb{1}_x(A)$

m is a measure: . $m(\emptyset) = \int_E \nu(\emptyset)\,d\mu = 0$

. $m(\bigcup_{n\in\mathbb{N}} M_n) = \int_E \nu((\bigcup_{n\in\mathbb{N}} M_n)_x) = \int_E \nu(\bigcup_{n\in\mathbb{N}}(M_n)_x) = \int \sum_{n\in\mathbb{N}} \nu((M_n)_x)\,d\mu(x)$

$= \sum_{n\in\mathbb{N}} \int \nu((M_n)_x)\,d\mu(x) = \sum_{n\in\mathbb{N}} m(M_n)$ (monotone convergence)

exemple: Lebesgue measure on $\mathbb{R}^d$; $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R})^{\otimes d}, \ell_d := \ell^{\otimes d})$

$$\lambda_d\left(\prod_{i=1}^d [a_i,b_i]\right) = \prod_{i=1}^d (b_i - a_i)$$

Theorem: Fubini-Tonelli (Fubini)

If μ, ν are σ-finite, and $f : (E \times F, \mathcal{A} \otimes \mathcal{B}) \to \overline{\mathbb{R}_+}$ ($\overline{\mathbb{R}}$) measurable (integrable)

then $x \mapsto \int_F f(x,y)\, d\nu(y)$ and $y \mapsto \int_E f(x,y)\, d\mu(x)$ are measurable (integrable) and

$$\int_{E\times F} f(x,y)\, d(\mu\otimes\nu)(x,y) = \int_E \left(\int_F f(x,y)\, d\nu(y) \right) d\mu(x) = \int_F \left(\int f(x,y)\, d\mu(x) \right) d\nu(y)$$

proof: Indicator: if $\Pi \in \mathcal{A}\otimes\mathcal{B}$, $y \mapsto \int_F \mathbb{1}_\Pi(x,y)\, d\nu(y) = \nu(\Pi_x)$, $y \mapsto \int \mathbb{1}_\Pi(x,y)\, d\mu(x) = \mu(\Pi^y)$ are measurable

and $\mu\otimes\nu(\Pi) = \int_E \nu(\Pi_x)\, d\mu(x) = \int_F \mu(\Pi^y)\, d\nu(y)$ from product measure construction.

simple function: linearity of the integral

$f : E\times F \to \overline{\mathbb{R}_+}$ measurable: let $s_n : E\times F \to \mathbb{R}_+$ a non decreasing sequence of simple functions with $s_n \xrightarrow[n\to+\infty]{\text{pointwise}} f$.

$$\int_{E\times F} f\, d\mu\otimes\nu \underset{\text{monotone convergence}}{=} \lim_{n\to+\infty} \int_{E\times F} s_n\, d\mu\otimes\nu = \lim_{n\to+\infty} \int_E \Big(\underbrace{\int_F s_n(x,y)\, d\nu(y)}_{:= h_n(x) \text{ increasing}} \Big) d\mu(x)$$

if $s_n = \sum_{i=1}^n \lambda_i \mathbb{1}_{\Pi_i}$, $h_n(x) = \sum_{i=1}^n \lambda_i \int_F \mathbb{1}_{\Pi_i}(x,y)\, d\nu(y) = \sum_{i=1}^n \lambda_i\, \nu((\Pi_i)_x)$ measurable

by monotone convergence $\int_{E\times F} f\, d\mu\otimes\nu = \int_E \lim_{n\to+\infty} h_n(x)\, d\mu(x) = \int_E \left(\int_F f(x,y)\, d\nu(y) \right) d\mu(x)$

by monotone convergence again, $\lim_{n\to+\infty} h_n(x) = \underbrace{\int_F f(x,y)\, d\nu(y)}$ measurable as a function of x.

$f : E\times F \to \overline{\mathbb{R}}$ integrable: by the Fubini-Tonelli theorem

$\int_{E\times F} |f|\, d\mu\otimes\nu = \int_E \left(\int_F |f(x,y)|\, d\nu(y) \right) d\mu(x) < +\infty$ so $x \to \int_F |f(x,y)|\, d\nu(x)$ is integrable

and $\int_{E\times F} f\, d\mu\otimes\nu = \int_{E\times F} f_+\, d\mu\otimes\nu - \int_{E\times F} f_-\, d\mu\otimes\nu = \int_E \left(\int_F f_+(x,y)\, d\nu(y) \right) d\mu(x) - \int_E \left(\int_F f_-(x,y)\, d\nu(y) \right) d\mu(x)$

$$\underset{\text{linearity of the } \mu \text{ integral}}{=} \int_E \left(\int_F f_+(x,y)\, d\nu(y) - \int_F f_-(x,y)\, d\nu(y) \right) d\mu(x) \underset{\text{linearity of the } \nu \text{ integral}}{=} \int_E \left(\int_F f(x,y)\, d\nu(y) \right) d\mu(x)$$

Change of variable:

def: . C^1 diffeomorphism: $\mathcal{O} \subset \mathbb{R}^d$, open, $\varphi : \mathcal{O} \to \varphi(\mathcal{O}) \in C^1(\mathcal{O})$ bijective of inverse C^1

. Jacobian matrix: $\varphi =: (\varphi_1, \dots, \varphi_d)$, $J_\varphi := \left(\frac{\partial \varphi_i}{\partial x_j} \right)_{i,j} = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial \varphi_n}{\partial x_1} & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{pmatrix}$

Jacobian determinant: $\det(J_\varphi)$

Theorem: change of variable (in $\mathbb{R}^d$)

Let $\varphi : O \subset \mathbb{R}^d \to \varphi(O)$ be a C^1-diffeomorphism, then $\forall f : \varphi(O) \to \mathbb{R}$ integrable,

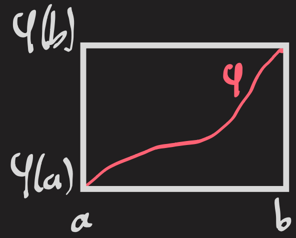
$$\int_O f \circ \varphi \, d\ell_d = \int_{\varphi(O)} f \left| \det J_{\varphi^{-1}} \right| d\ell_d$$

Remark:
- chain rule: $J_{\varphi^{-1}}(\varphi(x)) J_\varphi(x) = Id \Rightarrow \det J_\varphi(x)^{-1} = J_{\varphi^{-1}}(\varphi(x)) \neq 0$
- in $d=1$: φ C^1 diffeo $\Rightarrow$ φ strictly monotone

If f strictly increasing:

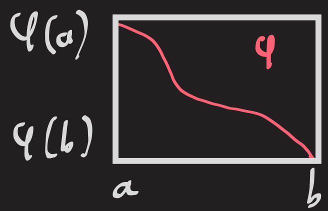
$$\int_a^b f(\varphi(x))\, dx = \int_{\varphi(a)}^{\varphi(b)} f(y) (\varphi^{-1})'(y)\, dy = \int_{\varphi(]a,b[)} f(y) |(\varphi^{-1})'(y)| dy$$

$y := \varphi(x)$ | $x = \varphi^{-1}(y)$, $dx = (\varphi^{-1})'(y)\, dy$



If f strictly decreasing:

$$\int_a^b f(\varphi(x))dx = \int_{\varphi(a)}^{\varphi(b)} f(y) (\varphi^{-1})'(y)\, dy = \int_{\varphi(]a,b[)} f(y)|(\varphi^{-1})'(y)| dy$$



- this is a specific case of $\int_O f \circ \varphi \, d\ell_d = \int_{f(O)} f \, d\varphi_* \ell_d$

Taking $f = \mathbb{1}_B$ for $B \in \mathcal{B}(\mathbb{R}^d)$, $B \subset \varphi(O)$,

$$\int_O \mathbb{1}_B \circ \varphi \, d\ell_d = \ell_d(\varphi^{-1}(B)) = \int_B |\det J_{\varphi^{-1}}| \, d\ell_d = \varphi_* \ell_d(B)$$

or with φ^{-1} instead of φ, $\ell_d(\varphi(B)) = \int_B |\det J_\varphi| \, d\ell_d$ for $B \subset \varphi^{-1}(O)$

volume modification induced by φ

- we prove the statement in the linear case $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is an isomorphism. It can be generalized using $\varphi(x + \varepsilon h) = \varphi(x) + \varepsilon J_\varphi(x) h + o(\varepsilon)$

proof: Let $\mu_\varphi(B) := \ell_d(\varphi(B))$, $\mu_\varphi(B + x) = \ell_d(\varphi(B+x)) = \ell_d(\varphi(B) + \varphi(x))$

$$= \ell_d(\varphi(B)) = \mu_\varphi(B)$$

and $\mu_\varphi([0,1]^d) = \ell_d(\varphi([0,1]^d)) < +\infty$ so $\mu_\varphi = \ell_d \; \lambda_d([0,1]^d)$

by unicity property of ℓ_d, ie $\ell_d(\varphi(B)) = \ell_d(B) \, \ell_d(\varphi[0,1]^d)$ (*)

$$\int_B |\det J_\varphi(x)| \, d\ell_d(x) = |\det \varphi| \ell_d(B)$$

we need to show $\ell_d(\varphi(B)) = |\det \varphi| \, \ell_d(B)$

then we conclude by linearity and monotone convergence.

$\ell_d(\varphi([0,1]^d)) = |\det \varphi|$

$\varphi = \varphi_1 \dots \varphi_k$ — permutation matrix: $\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots & 1 \end{pmatrix} =: \sigma_{ij}$

or

linear combination matrix $\begin{pmatrix} 1 & & \\ & \ddots & 1 \\ & & 1 \end{pmatrix} =: \tau_{i,j}$

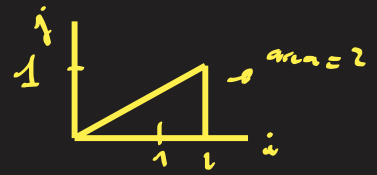
or

diagonal

by ⊛ and induction, $\ell_d(\varphi[0,1]^d) = \ell_d(A_1[0,1]^d) \dots \ell_1(A_k[0,1]^1)$

But:

- $\sigma_{ij}[0,1]^d = [0,1]^d$ so $\ell_d(\sigma_{ij}[0,1]^d) = 1 = |\det \sigma_{ij}|$
- $D = \begin{pmatrix} x_1 & (0) \\ (0) & x_d \end{pmatrix} \Rightarrow \ell_d(D[0,1]^d) = \ell_d\left(\prod_{i=1}^{d}[0,x_i]\right) = \prod_i |x_i| = |\det(D)|$
- $\tau_{i,j}[0,1]^d = \{(x_1, \dots, x_i + x_j, \dots, x_d),\ x_{1:d} \in [0,1]^d\}$



so $\ell_d(\tau_{i,j}[0,1]^d) = 1 = |\det \tau_{ij}|$

so $\ell_d(\varphi[0,1]^d) = |\det(A_1) \dots \det(A_k)| = |\det \varphi|$

9. L^p spaces

small ℓ^p spaces for sequences:

$p \in [1, +\infty[$, $\ell^p = \{u := (u_n)_{n\in\mathbb{N}} \subset \mathbb{R} \mid \sum_{n\in\mathbb{N}} |u_n|^p < +\infty\}$, if $u \in \ell^p$, $\|u\|_{\ell^p} := \left(\sum_{n\in\mathbb{N}} |u_n|^p\right)^{1/p}$

$p = \infty$, $\ell^\infty = \{u := (u_n)_{n\in\mathbb{N}} \subset \mathbb{R} \mid \sup_{n\in\mathbb{N}} |u_n| < +\infty\}$, if $u \in \ell^\infty$, $\|u\|_{\ell^\infty} := \sup_{n\in\mathbb{N}} |u_n|$

exemple: $u_n := \frac{1}{n}$, $\sum_{n\in\mathbb{N}^*} \frac{1}{n^p} < +\infty \iff p > 1$, so $(\frac{1}{n})_{n\in\mathbb{N}^*} \notin \ell^1$, $\in \ell^p \ \forall p > 1$ (including $p = +\infty$)

Let $(E, \mathcal{A}, \mu)$ be a measured space. Idea: group measurable functions inside normed vector spaces.

Def: $L^p(E, \mathcal{A}, \mu)$ spaces

Let $f : E \to \overline{\mathbb{R}}$ measurable, $\forall p \in [1, \infty[$, $\|f\|_{L^p} := \left(\int_E |f|^p d\mu\right)^{1/p}$

$(p = \infty)$ $\|f\|_{L^\infty} := \inf\{M > 0 \mid \mu(|f|^{-1}(]M, +\infty])) = 0\}$

(by convention $\inf \emptyset = \infty$)

$\forall p \in [1, \infty]$, $L^p(E, \mathcal{A}, \mu) := \{f : E \to \mathbb{R} \text{ measurable} \mid \|f\|_{L^p} < +\infty\}/\sim$ where $f \sim g$ if $f = g$ μ a.e.

remark: • L^p are well defined, $f = g$ μ a.e $\Rightarrow$ $\forall p \in [1, +\infty[$, $\|f\|_{L^p} = \|g\|_{L^p}$; $\forall M > 0$, $\mu(|f|^{-1}(]M, +\infty])) = \mu(|g|^{-1}(]M, +\infty]))$ so $\|f\|_{L^\infty} = \|g\|_{L^\infty}$

• we will identify L^p functions with their equivalence class

$f \in L^\infty \iff \exists M > 0 \mid \mu(|f|^{-1}(]M, +\infty])) = 0 \iff \exists M > 0 \mid |f| \leq M$ μ a.e., for exemple $\mathbb{1}_{\mathbb{Q}} \sim 0$ so $\|\mathbb{1}_{\mathbb{Q}}\|_{L^\infty} = \|0\|_{L^\infty} = 0$

$|f|^{-1}(]M, +\infty]) = \{x \in E \mid |f(x)| > M\}$

examples: • on $([0,1], \mathcal{B}([0,1]), \ell)$ $\int_0^1 |\frac{1}{x^\alpha}|^p dx = \int_0^1 \frac{1}{x^{\alpha p}} dx < +\infty \iff \alpha p < 1$

If $\alpha \leq 0$, $\forall p \in [1, +\infty]$, $x \mapsto \frac{1}{x^\alpha} \in L^p$, If $\alpha > 0$, $x \mapsto \frac{1}{x^\alpha} \in L^p$ iff $p < \frac{1}{\alpha}$

• on $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \#)$ $\int_{\mathbb{N}} f \, d\# = \sum_{n\in\mathbb{N}} f(n)$ so $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \#) = \ell^p$.

Property:

let $p \in [1, +\infty]$, $f, g \in L^p(E, \mathcal{A}, \mu)$, $\lambda \in \mathbb{R}$, $\|\lambda f\|_{L^p} = |\lambda| \, \|f\|_{L^p}$

if $p < +\infty$ $\|f + g\|_{L^p}^p \leq 2^{p-1}(\|f\|_{L^p}^p + \|g\|_{L^p}^p)$, $\|f+g\|_{L^\infty} \leq \|f\|_{L^\infty} + \|g\|_{L^\infty}$

hence $L^p(E, \mathcal{A}, \mu)$ is a vector space. Moreover $\|f\|_{L^p} = 0 \Rightarrow f = 0$

to show that $\|\cdot\|_{L^p}$, we must improve the triangular inequality: $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$

proof: • if $p < +\infty$, $\|\lambda f\|_{L^p} = \left(\int |\lambda f|^p\right)^{1/p} = \left(|\lambda|^p \int |f|^p\right)^{1/p} = |\lambda| \, \|f\|_{L^p} < +\infty$

if $p = +\infty$, if $\lambda = 0$, $\|0 f\|_{L^\infty} = 0 = 0 \cdot \|f\|_{L^\infty}$

if $\lambda \neq 0$, $\|\lambda f\|_{L^\infty} = \inf\{M > 0 \mid \mu(\{x \in E \mid |f|(x) \leq M/|\lambda|\}) = 0\}$

$= \inf\{|\lambda| \tilde{M} > 0 \mid \mu(\{x \in E \mid |f|(x) \leq \tilde{M}\}) = 0\} = |\lambda| \, \|f\|_{L^\infty}$

$\tilde{M} := M/|\lambda|$

• if $p < +\infty$, $\|f + g\|_{L^p}^p = \int |f + g|^p d\mu \leq 2^{p-1}(|f|^p + |g|^p) < +\infty$ so $f + g \in L^p$

$|\cdot|^p$ is convex: $\left(\frac{|f+g|}{2}\right)^p \leq \left(\frac{|f| + |g|}{2}\right)^p \leq \frac{|f|^p}{2} + \frac{|g|^p}{2}$

• if $p = +\infty$ $|f + g| \leq \|f\|_{L^\infty} + \|g\|_{L^\infty}$ μ a.e. so $f + g \in L^\infty$ and $\|f+g\|_{L^\infty} \leq \|f\|_{L^\infty} + \|g\|_{L^\infty}$

($\|f\|_{L^\infty} + \|g\|_{L^\infty} < +\infty$)

finally $\|f\|_{L^p} = 0 \Rightarrow f = 0$ μ a.e. so $f = 0_{L^p}$

theorem: Hölder's inequality

Let $p,q \in [1,+\infty] \mid \frac{1}{p}+\frac{1}{q}=1$, $f \in L^p$, $g \in L^q$, then $fg \in L^1$ and $\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$

remark: for $q=p=2$, $\int |fg|^2 d\mu \leq \int |f|^2 d\mu \int |g|^2 d\mu$ (Cauchy-Schwarz inequality)

proof: • If $p=+\infty, q=1$, $\|fg\|_{L^1} = \int_E |f(x)g(x)| d\mu(x) \leq \|f\|_{L^\infty} \int_E |g(x)| d\mu(x) = \|f\|_{L^\infty} \|g\|_{L^1}$

• If $p,q \neq 1,+\infty$, let $u := \ln(|f|^p)$, $v := \ln(|g|^q)$ so $|f|^p = e^u$, $|g|^q = e^v$

by convexity of $x \mapsto e^x$, $e^{\frac{u}{p}+\frac{v}{q}} \leq \frac{e^u}{p} + \frac{e^v}{q}$ ($\frac{1}{p}+\frac{1}{q}=1$) $\left(e^{tu+(1-t)v} \leq te^u + (1-t)e^v \text{ with } t=\frac{1}{p}, (1-t)=\frac{1}{q} \right)$

so $|fg| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q}$ and $\|fg\|_{L^1} \leq \frac{\|f\|^p_{L^p}}{p} + \frac{\|g\|^q_{L^q}}{q}$

if f or $g=0$ Hölder's inequality is just $0 \leq 0$, otherwise applying this to $\frac{f}{\|f\|_{L^p}}$ and $\frac{g}{\|g\|_{L^q}}$:

$$\frac{\|fg\|_{L^1}}{\|f\|_{L^p}\|g\|_{L^q}} \leq \frac{\left(\frac{\|f\|_{L^p}}{\|f\|_{L^p}}\right)^p}{p} + \frac{\left(\frac{\|g\|_{L^q}}{\|g\|_{L^q}}\right)^q}{q} = \frac{1}{p}+\frac{1}{q} = 1 \text{ so } \|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$$

Property: $\forall p \in [1,+\infty]$, $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$ (Minkowski inequality) and $L^p(E,\mathcal{A},\mu)$ is a normed vector space

proof: for $p=1$ it's the triangular inequaty in $\mathbb{R}$, we already obtained the result for $p=+\infty$. Let $p \in]1,+\infty[$,

$(f+g)^p = f(f+g)^{p-1} + g(f+g)^{p-1}$ so by Hölder's inequality for $\frac{1}{p}+\frac{1}{q}=1$:

$$\|f+g\|^p_{L^p} \leq \|f(f+g)^{p-1}\|_{L^1} + \|g(f+g)^{p-1}\|_{L^1} \leq \|f\|_{L^p}\|(f+g)^{p-1}\|_{L^q} + \|g\|_{L^p}\|(f+g)^{p-1}\|_{L^q}$$

$$= \left(\|f\|_{L^p} + \|g\|_{L^p}\right) \|f+g\|^{p-1}_{L^{q(p-1)}} \underset{*}{=} \left(\|f\|_{L^p} + \|g\|_{L^p}\right)\|f+g\|^{p-1}_{L^p}$$

$*$ $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow p+q=pq \Rightarrow q(p-1)=p$

if $\|f+g\|_{L^p} \neq 0,\infty$ we can conclude. If $\|f+g\|_{L^p}=0$ Minkowski's inequality is true
if $\|f+g\|_{L^p}=+\infty$, $\|f\|_{L^p}=+\infty$ or $\|g\|_{L^p}=+\infty$ by the previous statement.

Property: Countable triangular inequality

Let $(f_n)_{n\in\mathbb{N}} \subset L^p$. $\left\|\sum_{n\in\mathbb{N}} f_n\right\|_{L^p} \leq \sum_{n\in\mathbb{N}} \|f_n\|_{L^p}$

proof: i) $\left|\sum_{n\in\mathbb{N}} f_n(x)\right| \leq \sum_{n\in\mathbb{N}} |f_n(x)|$ gives $p=+\infty$, if $p<+\infty$

$$\left\|\sum_{n=1}^{+\infty} f_n\right\|_{L^p} = \left(\int\left|\sum_{n=0}^{+\infty} f_n\right|^p d\mu\right)^{1/p} \leq \left(\int\left(\sum_{n=0}^{+\infty} |f_n|\right)^p d\mu\right)^{1/p} = \lim_{k\to+\infty}\left(\int\left(\sum_{n=0}^{k} |f_n|\right)^p d\mu\right)^{1/p}$$

(monotone convergence)

$$= \lim_{k\to+\infty}\left\|\sum_{n=0}^{k}|f_n|\right\|_{L^p} \leq \lim_{k\to+\infty}\sum_{n=0}^{k}\||f_n|\|_{L^p} = \sum_{n\in\mathbb{N}}\|f_n\|_{L^p}$$

(induction on triangular $\leq$)

Theorem: (Riesz–Fischer)

$\forall p \in [1,+\infty]$, $L^p(E,\mathcal{A},\mu)$ is complete

proof: Let $(f_n)_{n\in\mathbb{N}} \subset L^p(E,\mathcal{A},\mu)$ be cauchy.

If $p=+\infty$: $|f_n(x)-f_m(x)| \leq \|f_n-f_m\|_{L^\infty}$ so $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy and $\mathbb{R}$ complete so $(f_n(x))_{n\in\mathbb{N}}$ converges

Let $f(x)=\lim_{n\to+\infty} f_n(x)$. $|f_n(x)-f(x)| = \lim_{p\to+\infty} \underbrace{|f_n(x)-f_{n+p}(x)|}_{\leq \|f_n-f_m\|_{L^\infty}} \leq \lim_{p\to+\infty} \|f_n-f_{n+p}\|_{L^\infty}$

so $\|f_n-f\|_{L^\infty} \leq \lim_{p\to+\infty} \|f_n-f_{n+p}\|_{L^\infty} \underset{n\to+\infty}{\longrightarrow} 0$ and $f\in L^\infty$ by triangular inequality.

if $p<+\infty$: Let $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ a subsequence st $\|f_{n_{i+1}}-f_{n_i}\|_{L^p} \leq \frac{1}{2^i}$

Let $g := \sum_{i=1}^{+\infty} (f_{n_{i+1}}-f_{n_i})$, $\|g\|_{L^p} \leq \sum_{i=1}^{+\infty} \|f_{n_{i+1}}-f_i\|_{L^p}^p = 1$ so g finite μ-a.e:

$\mu(g^{-1}(+\infty)) = \mu\left(\bigcap_{n\in\mathbb{N}} g^{-1}([n,+\infty])\right) = \lim_{n\to+\infty} \mu(g^{-1}([n,+\infty])) \leq \lim_{n\to+\infty} \frac{\int |g|}{n}$

Let $f := f_{n_1} + g = \lim_{i\to+\infty} f_{n_i}$, by Fatou's lemma:

$$\|f_n-f\|_{L^p}^p = \int_E |f_n-f|^p d\mu = \int_E \liminf_{i\to+\infty} |f_n-f_{n_i}|^p d\mu \leq \liminf_{i\to+\infty} \int_E |f_n-f_{n_i}|^p d\mu = \liminf_{i\to+\infty} \|f_n-f_{n_i}\|_{L^p}^p \underset{n\to+\infty}{\longrightarrow} 0$$

($(f_n)_n$ cauchy, $n_i \underset{i\to+\infty}{\longrightarrow} +\infty$)

and $f\in L^p$ by triangular inequality.

remark: we proved that $f_n \overset{L^p}{\underset{n\to+\infty}{\longrightarrow}} f \Rightarrow f_n \overset{\mu\text{-a.e}}{\underset{n\to+\infty}{\longrightarrow}} f$ up to a subsequence ($(f_n)_n$ cauchy)

Property: Embeddings

Let $p_1, p_2 \in [1,+\infty]$, $p_1 \leq p_2$,
1) if $\mu(E)<+\infty$, $\|\cdot\|_{L^{p_1}} \leq \mu(E)^{1/p_1 - 1/p_2} \|\cdot\|_{L^{p_2}}$ so $L^{p_2} \subset L^{p_1}$
2) $\|\cdot\|_{\ell^{p_2}} \leq \|\cdot\|_{\ell^{p_1}}$ so $\ell^{p_1} \subset \ell^{p_2}$
3) interpolation: if $\frac{1}{r} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$, $\|\cdot\|_{L^r} \leq \|\cdot\|_{L^{p_1}}^{\alpha} \|\cdot\|_{L^{p_2}}^{1-\alpha}$ so $p_1\leq r\leq p_2 \Rightarrow L^{p_1}\cap L^{p_2} \subset L^r$

remark: the injections $L^{p_2} \to L^{p_1}$, $f \mapsto f$; $\ell^{p_1} \to \ell^{p_2}$, $f\mapsto f$ are continuous

proof: 1) Let f measurable, $\|f\|_{L^{p_1}}^{p_1} = \|f^{p_1}\|_{L^1} \leq \|1\|_{L^q} \|f^{p_1}\|_{L^p} = \mu(E)^{1/q} \|f\|_{L^{pp_1}}^{p_1}$ (Hölder's inequality)

take $p := \frac{p_2}{p_1} \geq 1$, $\frac{1}{q} = 1 - \frac{1}{p} = 1 - \frac{p_1}{p_2}$ so $\|f\|_{L^{p_1}} \leq \mu(E)^{\frac{1}{p_1}-\frac{1}{p_2}} \|f\|_{L^{p_2}}$

2) Let $u := (u_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, $\forall n\in\mathbb{N}$, $\frac{u_n}{\|u\|_{p_1}} \leq 1$ so $\frac{u_n^{p_2}}{\|u\|_{\ell^{p_1}}^{p_2}} \leq \frac{u_n^{p_1}}{\|u\|_{\ell^{p_1}}^{p_1}}$. Summing on n, $\frac{\|u\|_{\ell^{p_2}}^{p_2}}{\|u\|_{\ell^{p_1}}^{p_2}} \leq 1$

so $\|u\|_{\ell^{p_2}} \leq \|u\|_{\ell^{p_1}}$

3) $p_1\leq r\leq p_2 \Rightarrow \exists\, 0\leq\alpha\leq 1 \mid \frac{1}{r} \underset{(1)}{=} \frac{\alpha}{p_1} + \frac{(1-\alpha)}{p_2}$

$\|f\|_{L^r}^r = \|f^r\|_{L^1} \underset{\text{Hölder},\ \frac{1}{p}+\frac{1}{q}:=1\ (2)}{\leq} \|f^{r\alpha}\|_{L^p} \|f^{r(1-\alpha)}\|_{L^q} = \|f\|_{L^{r\alpha p}}^{r\alpha} \|f\|_{L^{r(1-\alpha)q}}^{r(1-\alpha)}$ ($r\alpha p = p_1$)

Fix $p := \frac{p_1}{\alpha r}$ (3) so $r\alpha p = p_1$ and $r(1-\alpha)q \underset{(1)}{=} \left(1-\frac{r\alpha}{p_1}\right)p_2 q \underset{(3)}{=} \left(1-\frac{1}{p}\right) q p_2 \underset{(2)}{=} p_2$ so $\|f\|_{L^r} \leq \|f\|_{L^{p_1}}^{\alpha} \|f\|_{L^{p_2}}^{(1-\alpha)}$

Remark: $\|u\|_{\ell^p} \leq \|u\|_{\ell^1}$ so $\sum_{n\in\mathbb{N}} |u_n|^p \leq \left(\sum_{n\in\mathbb{N}} |u_n|\right)^p$

Duality

Def: dual space:

Let E be a Banach space (complete normed vector space) $E^* := \{ \varphi \in C^0(E, \mathbb{R}) \text{ linear} \}$ (linear form), $\|\varphi\|_{E^*} := \sup_{\|x\|_E \leq 1} \|\varphi x\|_E$

Remark: $p^* := \frac{p}{p-1}$ so $\frac{1}{p} + \frac{1}{p^*} = 1$ then if $u \in L^{p^*}$, $\varphi_u : L^p \to \mathbb{R}$, $f \mapsto \int uf$ is linear and $\|\varphi_u(f)\| \leq \|u\|_{L^{p^*}} \|f\|_{L^p}$ (continuity for linear functions)

so $\varphi_u \in (L^p)^*$

$$\|\varphi_u\|_{(L^p)^*} = \|u\|_{L^{p^*}}$$

$\boxed{\leq}$ $\|\varphi_u\|_{(L^p)^*} = \sup_{\|f\|_{L^p} \leq 1} \left| \int uf \right| \leq \|u\|_{L^{p^*}}$ ($\leq$ Hölder)

$\boxed{\geq}$ taking $f := \begin{cases} u^{p^*-1} & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$, $\|f\|_{L^p}^p = \int |u|^{(p^*-1)p} d\mu = \|u\|_{L^{p^*}}^{p^*}$ and $\frac{|\int uf|}{\|f\|_{L^p}} = \|u\|_{L^{p^*}}^{p^* - \frac{p^*}{p}} = \|u\|_{L^{p^*}}$

so $L^{p^*} \to (L^p)^*$, $u \mapsto \varphi_u$ is isometric, injective, linear

theorem: Riesz representation theorem (for L^p spaces)

Let $p \in]1, +\infty[$, $\varphi \in (L^p)^*$ then $\exists! \, u_\varphi \in L^{p^*}$ such that $\forall f \in L^p$, $\varphi(f) = \int u_\varphi f$.
Moreover $\|\varphi\|_{(L^p)^*} = \|u_\varphi\|_{L^{p^*}}$. For $p = 1$ the conclusion hold for σ-finite μ.

Remark:
- $(L^p)^* \simeq L^{p^*}$ for $p \neq +\infty$,
- for $p = +\infty$, $\ell^1 \not\simeq (\ell^\infty)^*$: $h : \ell^\infty \to \mathbb{R}$, $(x_n)_n \mapsto \lim_{n \to +\infty} x_n$ (extended by Hahn Banach theorem) $|h(x_n)_n| \leq \|(x_n)_n\|_{\ell^\infty}$ so $h \in (\ell^\infty)^*$ but

If $h(x_n)_{n \in \mathbb{N}} = \sum_{n \in \mathbb{N}} y_n x_n$ for $(y_n) \in \ell^1$, $h\,\delta_n = y_n = 0$ so $h = 0$ absurd because $h(1, \ldots) = 1$ ($\in \ell^\infty$), where $\delta_n = (0, \ldots, 0, 1, 0 \ldots)$ (1 at position n)

- L^2 is a Hilbert space (Banach space where the norm comes from a $\langle \cdot \rangle$): $\langle f, g \rangle_{L^2} = \int_E fg \, d\mu$ is a scalar product (well defined by Cauchy-Schwarz inequality $\langle f, g \rangle_{L^2} \leq \|f\|_{L^2} \cdot \|g\|_{L^2}$)
  with $\langle f, f \rangle_{L^2} = \|f\|_{L^2}^2$
- Riesz representation theorem for Hilbert spaces: $H^* \simeq H$ i.e. $\forall \varphi \in H^* \; \exists x_\varphi \in H$ st $\forall y \in H$ $\varphi(y) = \langle x_\varphi, y \rangle$

Fourier Series

$e_k := \frac{1}{\sqrt{2\pi}} e^{ikx}$ orthonormal in $L^2([0,2\pi],\mathbb{C})$ with $k \in \mathbb{Z}$.

Def: Hilbert space and Hilbert basis. , **Prop:** Parseval's identity and Bessel's inequality:

Bessel's inequality follows from the identity

$$0 \le \left\| x - \sum_{k=1}^{n} \langle x, e_k\rangle e_k \right\|^2 = \|x\|^2 - 2\sum_{k=1}^{n} \mathrm{Re}\langle x, \langle x, e_k\rangle e_k\rangle + \sum_{k=1}^{n} |\langle x, e_k\rangle|^2$$
$$= \|x\|^2 - 2\sum_{k=1}^{n} |\langle x, e_k\rangle|^2 + \sum_{k=1}^{n} |\langle x, e_k\rangle|^2$$
$$= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2,$$

Remark: : algebraic basis not countable and $\ell^2(\mathbb{C})$.

th: Stone – Weierstrass

Let (X,d) be a compact metric space and $S \subset X$ a sub-algebra (unitary) that separates X: $\forall x \neq y \in X, \exists f \in S \mid f(x) \neq f(y)$.
then S dense in $(X, \|\cdot\|_\infty)$ $\mathbb{C}$ valued: add closed under conjugaison.

ex: polynomials.

th: $C^0([0,2\pi],\mathbb{C})$ dense in $L^p([0,2\pi],\mathbb{C})$ $\forall p \in [1,+\infty[$

proof: $C^0(X)$ dense in $L^p(X)$ with $X \subseteq \mathbb{R}$ compact.

Let $B \in \mathcal{B}(K)$, by regularity ℓ, $\forall \varepsilon > 0$, $\exists U$ open, K compact st $K \subseteq B \subseteq U$ and $\ell(U \setminus K) < \varepsilon$

By Urysohn lemma, $\exists f \in C^0(X) \mid f_{|K} = 1$ and $0 \le f \le 1$, $\mathrm{supp}\, f \subseteq U$, hence,

$\int_X |\mathbb{1}_B - f|^p d\ell = \int_{U\setminus K} |\mathbb{1}_B - f|^p d\mu \le \varepsilon$, generalized by linearity and monotone convergence th.

$\square$

theorem: Fourier Series - decomposition:

Let $f \in L^2([0,2\pi],\mathbb{C})$, then $f = \sum_{k\in\mathbb{Z}} c_k e_k$ with $c_k = \langle e_k, f\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-ikx} dx$

this proof is very general and works for other decompositions (like wavelets)

proof: Let $P_{trigo} := \{ \sum_{k=-N}^{N} c_k e_k,\ N \in \mathbb{N},\ (c_k)_{k=-N:N} \subseteq \mathbb{C} \}$

by Stone-Weierstrass, $\overline{P_{trigo}}^{\|\cdot\|_\infty} = C^0([0,2\pi],\mathbb{C})$, taking closure in L^2 we get:

$$\overline{\overline{\mathcal{P}_{trigo}}^{\|\cdot\|_\infty}}^{\|\cdot\|_{L^2}} = \overline{C^0([0,2\pi],\mathbb{C})}^{\|\cdot\|_{L^2}} = L^2([0,2\pi],\mathbb{C})$$

$\overline{\mathcal{P}_{trigo}}^{\|\cdot\|_{L^2}}$ $\Vert$:

- $\boxed{\cup}$ $\mathcal{P}_{trigo} \subseteq \overline{\mathcal{P}_{trigo}}^{\|\cdot\|_\infty}$ so $\overline{\mathcal{P}_{trigo}}^{\|\cdot\|_{L^2}} \subseteq \overline{\overline{\mathcal{P}_{tri}}^{\|\cdot\|_\infty}}^{\|\cdot\|_{L^2}}$
- $\boxed{\cap}$ $\overline{\mathcal{P}_{trigo}}^{\|\cdot\|_\infty} \subseteq \overline{\mathcal{P}_{trigo}}^{\|\cdot\|_{L^2}}$ as $\|\cdot\|_{L^2} \leqslant 2\pi \|\cdot\|_{L^\infty}$ (cvg in $L^\infty \Rightarrow$ cvg in L^2)

  so $\overline{\overline{\mathcal{P}_{tri}}^{\|\cdot\|_\infty}}^{\|\cdot\|_{L^2}} \subseteq \overline{\overline{\mathcal{P}_{trigo}}^{\|\cdot\|_{L^2}}}^{\|\cdot\|_{L^2}} = \overline{\mathcal{P}_{trigo}}^{\|\cdot\|_{L^2}}$

Let $S_{trigo} := \left\{ \sum_{k\in\mathbb{Z}}' c_k e_k,\ (c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}) \right\}$

<u>We prove that $\overline{\mathcal{P}_{trigo}}^{\|\cdot\|_{L^2}} = S_{trigo}$:</u>

$\boxed{\supseteq}$ Let $(c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$, $\Big\| \sum_{k\in\mathbb{Z}}' c_k e_k - \underbrace{\sum_{k=-N}^{N}{}' c_k e_k}_{\in \mathcal{P}_{trigo}} \Big\|_{L^2}^2 = \Big\| \sum_{|k|>N}' c_k e_k \Big\|_{L^2}^2 \overset{\text{Bessel}}{\leqslant} \sum_{|k|>N}' \| c_k e_k \|_{L^2}^2$

$= \sum_{|k|>N}' |c_k|^2 \underset{N\to+\infty}{\longrightarrow} 0$ as $\sum_{k\in\mathbb{Z}}' |c_k|^2 < +\infty$. So $\sum_{k\in\mathbb{Z}}' c_k e_k \in \overline{\mathcal{P}_{trigo}}^{\|\cdot\|_{L^2}}$

$\boxed{\subseteq}$ First, $\mathcal{P}_{trigo} \subseteq S_{trigo}$, we conclude by proving that S_{trigo} is closed in L^2:

Let $f \in L^2$ and $(f_n)_{n\in\mathbb{N}} \subseteq S_{trigo}$ such that $f_n \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} f$. <u>Goal:</u> $f \in S_{trigo}$

$\forall n \in \mathbb{N}$, $\exists (c_{k,n})_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}) \mid f_n = \sum_{k\in\mathbb{Z}}' c_{k,n} e_k$. Noticing that

$\|f_n\|_{L^2}^2 \underset{\text{Parseval}}{=} \sum_{k\in\mathbb{Z}}' |c_{k,n}|^2 = \|(c_{k,n})_{k\in\mathbb{Z}}\|_{\ell^2(\mathbb{C})}^2$, as $(f_n)_{n\in\mathbb{N}}$ is Cauchy, so is $((c_{k,n})_{k\in\mathbb{Z}})_{n\in\mathbb{N}}$

hence $\exists (c_k)_{k\in\mathbb{Z}} \mid (c_{k,n})_{k\in\mathbb{Z}} \overset{\ell^2(\mathbb{C})}{\longrightarrow} (c_k)_{k\in\mathbb{Z}}$ and $\sum' c_{k,n} e_k \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} \sum_{k\in\mathbb{Z}}' c_k e_k$

$$\underset{f_n \underset{n\to+\infty}{\overset{L^2}{\longrightarrow}} f}{\Vert}$$

So, $f = \sum_{k\in\mathbb{Z}}' c_k e_k \in S_{trigo}$.

<u>Conclusion:</u> It follows that $S_{trigo} = L^2([0,2\pi],\mathbb{C})$. Let $f \in L^2([0,2\pi],\mathbb{C})$, then $\exists (c_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C}) \mid f = \sum_{k\in\mathbb{Z}}' c_k e_k$. Let $k \in \mathbb{Z}$, $\langle e_k, \cdot \rangle : L^2 \to L^2$ is continuous by Cauchy-Schwarz so $\langle e_k, f \rangle = c_k$ and $f = \sum_{k\in\mathbb{Z}}' \langle e_k, f \rangle$ $\square$

Fourier transform:

same than Fourier series except that the frequency is not quantized (no discrete values of $k \in \mathbb{Z}$) $\to k \in \mathbb{R}$

Def $\quad F: L^1 \to L^\infty$

$f \in L^1(\mathbb{R}, \mathbb{C})$, for $\omega \in \mathbb{R}$, $F(f)(\omega) := \hat{f}(\omega) = \langle e_\omega, f \rangle$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega \cdot x} dx$$

$$F^{-1}(f)(x) := \frac{1}{\sqrt{2\pi}} \int f(\omega) e^{i\omega \cdot x} d\omega$$

goal: $f = \int_{\mathbb{R}} \hat{f}(\omega) e_\omega d\omega$ ie $f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\omega) e^{i\omega \cdot x} dk = F^{-1}(F(f))(x)$

inverse formula

Plancherel: $\|f\|_{L^2}^2 = \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega = \|\hat{f}\|_{L^2}^2$, i.e. F is an isometry $L^2 \to L^2$.

orthonormality of $(e_\omega)_{\omega \in \mathbb{R}}$.

$$\text{"} \langle e_\omega, e_v \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix \cdot (v - \omega)} dx = \frac{1}{\sqrt{2\pi}} F(1)(v - \omega) \text{"}$$

$F(\mu)(\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega \cdot x} d\mu(x)$ defined if $\mu(\mathbb{R}) = +\infty$, dx

$F(\mu)$ defined as a measure as $F(\mu)(B) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\int_B e^{-i\omega \cdot x} d\omega \right) d\mu(x)$

$$= \int_{\mathbb{R}} \hat{\mathbb{1}}_B(x) d\mu(x) \qquad \text{"} = \int_B F(\mu)(\omega) d\omega \text{"}$$

then

$$\frac{1}{\sqrt{2\pi}} F(1)(B) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\mathbb{1}}_B(x) dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\mathbb{1}}_B(\omega) e^{i\omega \cdot 0} d\omega$$

$1 \cdot \ell$

(to a ≥ 0 function here $\ell = 1$ associate $B \to \int_B f(x) dx$)

$$= F^{-1}(F(\mathbb{1}_B))(0) = \mathbb{1}_B(0) = \delta(B) \quad \text{so } \frac{1}{\sqrt{2\pi}} F(1) = \delta$$

so $\langle e_\omega, e_v \rangle = \delta(\omega - v) := \delta_{\omega - v}$

prop: $f \in L^1$, $\hat{f}$ is C^0 and $\mathcal{F} : L^1 \to L^\infty$ is continuous

proof . Let $x \in \mathbb{R}$, $\omega \to e^{i\omega x}$ uniformly continuous so $\exists \eta$ modulus of C^0:
$|e^{i\omega x} - e^{i(\omega+\varepsilon)x}| \leq \eta(\varepsilon)$

then $|\hat{f}(\omega+\varepsilon) - \hat{f}(\omega)| \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x)| \eta(\varepsilon)\, dx \leq \frac{\eta(\varepsilon)}{\sqrt{2\pi}} \|f\|_{L^1} \xrightarrow[\varepsilon \to 0]{} 0$

. $|\hat{f}(\nu)| \leq \|f\|_{L^1}$ so $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$. □

Prop: Multiplication formula

if $f, g \in L^1$, $\int_{\mathbb{R}} \hat{f}(x) g(x)\, dx = \int_{\mathbb{R}} f(x) \hat{g}(x)\, dx$

proof: integral ✓ By L^1-L^∞ Hölder, By Fubini

Properties: . linearity

. $\mathcal{F}(f(\cdot - x_0))(\nu) = e^{-ix_0\nu} \hat{f}(\nu)$

. $\mathcal{F}(e^{i\omega \cdot} f)(\nu) = \hat{f}(\nu - \omega)$

. convolution $f * g(x) = \int_{\mathbb{R}} f(y) g(x-y)\, dy$

$\widehat{f * g} = \sqrt{2\pi}\, \hat{f}\, \hat{g}$

. $f, f' \in L^1 \Rightarrow \hat{f'}(\nu) = i\nu \hat{f}(\nu)$

$f, x f(x) \in L^1 \Rightarrow \hat{f}'(\nu) = -i \mathcal{F}(x \to x f(x))(\nu)$

proof: . $\mathcal{F}(f(\cdot - x_0))(\nu) = \frac{1}{\sqrt{2\pi}} \int f(x - x_0) e^{-ix\nu}\, dx = e^{-ix_0\nu} \hat{f}(\nu)$

$\mathcal{F}(e_k f)(\nu) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} f(x) e^{-i\nu x}\, dx = \hat{f}(\nu - k)$

. Fubini $\iint f(y) g(x-y) e^{-i\nu x}\, dx\, dy = \iint f(y) g(x) e^{-i\nu x - i\nu y}\, dx\, dy$

. $\hat{f'}(\nu) = \int f'(x) e^{-i\nu x}\, dx \underset{\text{IPP}}{=} -i\nu \hat{f}(\nu)$

$\lim_{x \to +\infty} f(x) = f(a) + \int_a^{+\infty} f'(x)\, dx \in \mathbb{R}$ and $= 0$ since $f \in L^1$.

. dominated cvg as $|x f(x) e^{-i\nu x}| \leq |x f(x)|$ integrable □

ex:

- $\mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\right) = \delta$ so $\mathcal{F}(e_\upsilon) = \delta_\upsilon$ (only frequency in e_υ is υ).

Lemma: $g_a(x) = \frac{1}{\pi^{1/4}\sqrt{a}} e^{-\frac{1}{2}\left(\frac{x}{a}\right)^2} \Rightarrow \hat{g_a} = g_{a^{-1}}$, $\|g_a\|_{L^2} = 1$ (inversion of length)

$a = 1 \to \mathcal{F}g_1 = g_1$

$$\|e^{-c\left(\frac{x}{a}\right)^2}\|_{L^2}^2 = \int e^{-2c\left(\frac{x}{a}\right)^2} dx = \left(2\pi \int r e^{-2c\left(\frac{r}{a}\right)^2} dr\right)^{1/2} = \frac{a}{\sqrt{2c}}\left(\pi \int 2 e^{-r^2} dr\right)^{1/2}$$

$$= a\sqrt{\frac{\pi}{2c}}$$

proof:

$$\begin{cases} \hat{g_a}'(\upsilon) = -i\,\mathcal{F}(x \mapsto x\, g_a(x))(\upsilon) \\ \widehat{g_a'}(\upsilon) = i\upsilon\, \hat{g_a}(\upsilon) \\ g_a'(x) = \frac{-x}{a^2} g_a(x) \end{cases}$$

so $\hat{g_a}'(\upsilon) = i a^2 \mathcal{F}(g_a')(\upsilon) = -\upsilon a^2 \hat{g_a}(\upsilon)$ and $\hat{g_a}(\upsilon) = c\, e^{-\frac{1}{2}(\upsilon a^2)}$

but $\hat{g_a}(0) = c = \frac{1}{\sqrt{2\pi}} \int g_a(x)\, dx = \frac{1}{\pi^{1/4}\sqrt{2\pi a}} \int e^{-1/2\left(\frac{x}{a}\right)^2} dx$

$$= \frac{\sqrt{a}}{\pi^{1/4}} \left(\int r e^{-r^2/2} dr\right)^{1/2} = \frac{\sqrt{a}}{\pi^{-1/4}} \quad \square$$

if we know the multiplication formula for measure: $\mathcal{F}^{-1}\mathcal{F} f(x) = \int_{\mathbb{R}} \frac{e^{ix\cdot\upsilon}}{\sqrt{2\pi}} \hat{f}(\upsilon)\, d\upsilon = \int_{\mathbb{R}} f(y)\, d\delta_x(y) = f(x)$

Idea: we use $g_a^2(\cdot - x)$ as $a \to 0$ as approximation for δ_x

Computation of a Fourier transform:

Let $h_{a,x}(\upsilon) := \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{a\upsilon}{2}\right)^2 + i\upsilon x}$, we have that, $\hat{h}_{a,x}(y) = g_a^2(y-x) = \frac{1}{\sqrt{\pi}a} e^{-\left(\frac{y-x}{a}\right)^2}$

$$h_{a,x}(y) = \frac{1}{\sqrt{2}\pi^{1/4}} \cdot \frac{1}{\pi^{1/4}} \cdot \sqrt{\frac{a}{\sqrt{2}}} \cdot \sqrt{\frac{\sqrt{2}}{a}}\, e^{-\frac{1}{2}\left(\frac{a\upsilon}{\sqrt{2}}\right)^2 + i\upsilon x} = \frac{1}{(2\pi)^{1/4}\sqrt{a}}\, g_{\frac{\sqrt{2}}{a}}(\upsilon)\, e^{i\upsilon x}$$

(where $e^{-\frac{1}{2}\left(\frac{a\upsilon}{\sqrt{2}}\right)^2}$ term gives $g_{\frac{\sqrt{2}}{a}}(\upsilon)$)

so $\hat{h}_{a,x}(y) = \frac{1}{(2\pi)^{1/4}\sqrt{a}}\, g_{\frac{a}{\sqrt{2}}}(y-x) = \frac{1}{(2\pi)^{1/4}\sqrt{a}} \cdot \frac{2^{1/4}}{\pi^{1/4}\sqrt{a}} e^{-\left(\frac{y-x}{a}\right)^2} = g_a^2(y-x)$

property: translation continuity in L^p, $p \in [1, +\infty[$

let $f \in L^p$, then $\begin{array}{l} \mathbb{R} \to L^1 \\ \varepsilon \mapsto \tau_\varepsilon f : x \mapsto f(x-\varepsilon) \end{array}$ is continuous

proof: If $h \in C_c^\infty$ with support K then $\|h - \tau_\varepsilon h\|_{L^p}^p = \int_{K \cup (K+\varepsilon)} |f(x-\varepsilon) - f(x)|^p dx \leq \underbrace{\eta(\varepsilon)^p}_{\text{modulus of } C^0 \text{ of } f} \overbrace{2\ell(K)}^{<+\infty} \underset{\varepsilon \to 0}{\to} 0$

let $\underset{\in C_c^\infty}{f_n} \underset{n \to +\infty}{\overset{L^1}{\to}} f$,

$$\|f - \tau_\varepsilon f\|_{L^p} \leq \|f - \underset{\in C_c^\infty}{f_n}\|_{L^p} + \|f_n - \tau_\varepsilon f_n\|_{L^p} + \|\tau_\varepsilon f_n - \tau_\varepsilon f\|_{L^p}$$

$$= \underbrace{2\|f - f_n\|_{L^p}}_{\underset{n \to +\infty}{\to} 0 \text{ uniformly in } \varepsilon} + \underbrace{\|f_n - \tau_\varepsilon f_n\|_{L^p}}_{\underset{\varepsilon \to 0}{\to} 0 \text{ for a given } n}$$

□

property: Minkowsky's integral inequality (Should be in the L^p spaces chapter)

$f : (E, \mathcal{A}, \mu) \times (F, \mathcal{B}, \nu) \to \mathbb{C}$, $\left\| \int_E f(x, \cdot) \, d\mu(x) \right\|_{L^p(F)} \leq \int_E \|f(x, \cdot)\|_{L^p(F)} d\mu(x)$

proof: $F := \int_E f(x, \cdot) \, d\mu(x)$, $\|F\|_{L^p(F)} \underset{\text{Riesz}}{=} \sup_{\substack{g \in L^q(F) \\ \|g\|_{L^q(F)} = 1}} \|Fg\|_{L^1(F)}$ Let $g \in L^q(F)$, $\|g\|_{L^q(F)} = 1$

$$\|Fg\|_{L^1(F)} \leq \int_F \int_E |f(x,y) g(y)| \, d\mu(x) \, d\nu(y) \underset{\text{Fubini}}{=} \int_E \|f(x, \cdot) g\|_{L^1(F)} d\mu(x) \underset{\text{Holder}}{\leq} \int_E \|f(x, \cdot)\|_{L^p(F)} d\mu(x)$$

□

Lemma: regularization

let $f \in L^p$, $\underbrace{g_a^2 * f}_{C^\infty} \underset{a \to +\infty}{\overset{L^p}{\to}} f$

proof: $g_a^2 * f(x) - f(x) = \int_{\mathbb{R}} g_a^2(y) \left(f(x-y) - f(x) \right) dy = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} |f(x - ay) - f(x)| \, dy$

so by Minkowsky's integral inequality

$$\|g_a^2 * f - f\|_{L^p} \leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \overbrace{\underbrace{\|f(\cdot - ay) - f\|_{L^p}}_{\underset{a \to 0}{\overset{\text{ptw}}{\to}} 0 \text{ by translation continuity in } L^p}}^{\leq 2\|f\|_{L^p} e^{-y^2} \text{ integrable independent of } a} \underset{a \to 0}{\to} 0 \text{ by dominated convergence}$$

□

theorem: inversion on L^1

if $f, \hat{f} \in L^1$ then $\mathcal{F}^{-1}\mathcal{F} f = f$

proof:

multiplication formula: let $x \in \mathbb{R}$, $\int_{\mathbb{R}} f(y)\, g_a^2(y-x)\, dy = \int_{\mathbb{R}} \hat{f}(\upsilon)\, h_{a,x}(\upsilon)\, d\upsilon$

$(g_a^2 * f)(x) \xrightarrow{L^1} f$ | $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\upsilon) e^{i\upsilon x} e^{-\left(\frac{a\upsilon}{2}\right)^2} d\upsilon \xrightarrow[a\to 0]{} \mathcal{F}^{-1}\mathcal{F} f(x)$

by dominated cvg with $\hat{f} \in L^1$. □

theorem:

if $f \in L^1 \cap L^2$ then $\hat{f} \in L^2$ and $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$

proof: let $g = \bar{f}(-\cdot)$ as $\hat{g}(\upsilon) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \bar{f}(-x) e^{-i\upsilon x} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \bar{f}(x) e^{i\upsilon x} dx = \overline{\frac{1}{\sqrt{2\pi}} \int f(x) e^{-i\upsilon x} dx} = \bar{\hat{f}}$

$\hat{\hat{g}}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \bar{\hat{f}}(\upsilon) e^{-i\upsilon x} dx = \overline{\frac{1}{\sqrt{2\pi}} \int \hat{f}(\upsilon) e^{i\upsilon x} dx} = \overline{f(x)}$

by the multiplication formula: $\int |\hat{f}|^2 = \int \hat{g} \hat{f} = \int \hat{\hat{g}} f = \int |f|^2$ □

Corollary: $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$ if $f, g \in L^2$.

theorem: $\mathcal{F}$ can be extended on L^2 by continuity and the Plancherel and inverse theorem hold for $\mathcal{F}: L^2 \to L^2$, same for $\mathcal{F}^{-1}$

proof: We obtained that $\mathcal{F}: (L^1 \cap L^2, \|\cdot\|_{L^2}) \to L^2$ is C^0 as $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$

$L^1 \cap L^2$ is dense in L^2 so $\mathcal{F}$ can be extended by continuity on L^2.

as $\mathcal{F}: L^2 \to L^2$ and $\|\cdot\|_{L^2}$ are continuous Plancherel formula remains true for $\mathcal{F}: L^2 \to L^2$.

The same holds for $\mathcal{F}^{-1}$ as $\mathcal{F}^{-1}(f)(\upsilon) = \mathcal{F}(f)(-\upsilon)$

$g_a^2 * f \in L^1 \cap L^2$ and $\widehat{g_a^2 * f} = \underbrace{\hat{g_a^2}}_{\in L^2} \underbrace{\hat{f}}_{\in L^2} \in L^1$

by the L^1 inverse Fourier theorem,

$\mathcal{F}^{-1}\mathcal{F}\, g_a^2 * f = g_a^2 * f \xrightarrow[a\to 0]{L^2} f$ so $\mathcal{F}^{-1}\mathcal{F} f = f$ □

$\downarrow$ ($a \to 0$; $\mathcal{F}^{-1}, \mathcal{F}$ C^0 on L^2)

$\mathcal{F}^{-1}\mathcal{F} f$