# Real Analysis: Measure Theory - Lecture notes

School of science - Bachelor of mathematics
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# Introduction

## Related readings

For further references related to the course material we refer to [3] for its vast collection of problems and solutions. [1] is also a classic, we will cover chapters I, II, VI and VIII. We also must mention that these lecture notes are inspired from the great ones of J-C. Breton [2] (in French).

## Let's measure

As its name subtly suggests, measure theory is a framework for measuring the 'size' of objects. 'What kind of object?' the audience should ask. To that, the answer is quite rude and abstract: sets. This is the reason why you will need some familiarity with set theory and quantifiers to understand the following piece of mathematics.

But be not afraid (yet), we will start by measuring length of segments, areas of rectangles and volumes. For these familiar objects, intuition about 'size' will guide us through the construction of the abstract theory of measures. Once established, this theory extends to the study of more complex structures, such as fractals.

Measure theory is in particular renowned for its application to integration. The central object of this course will likely be the Lebesgue integral, which, under a regularity assumption, generalizes the Riemann integral. The case for the Lebesgue integral rests on the following idea: if we have a notion of 'size' on an arbitrary set $E$ (that is, a theory for measuring objects) then we can define the integral of some functions whose domain is $E$. The key advantage is that, unlike the Riemann integral, $E$ can differ significantly from $\mathbb{R}$.

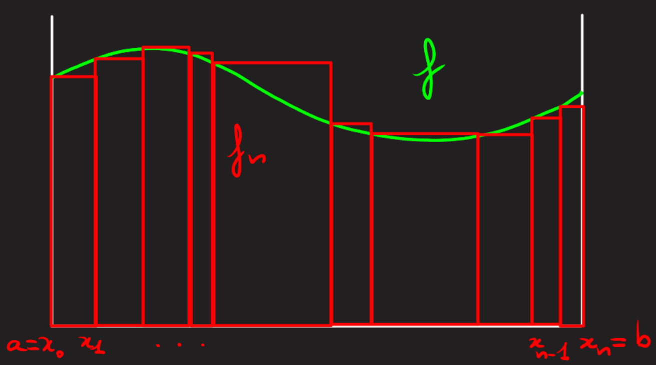
Let us develop on the link between a theory of integration (understand here: a way of defining integrals) and a notion of 'size' while briefly recalling Riemann's theory of integration.

## Riemann integral in a nutshell

Let $f$ be a real function on an interval $[a, b]$ that we assume continuous for simplicity. Let $(x_k)_{k\in[\![0,n]\!]}$ be a partition of $[a, b]$ and consider the piecewise constant fonction

$$f_n(x) \coloneqq f(x_k) \text{ if } x_k \leqslant x < x_{k+1}$$

From our great knowledge of the formula for the area of rectangles we are able to define the integral of $f_n$ as




$$I_n \coloneqq \int_a^b f_n(x)dx \coloneqq \sum_{k=0}^{n-1} f(x_k)(x_{k+1} - x_k)$$

Then, imposing the partitions are of increasing precision:

$$\epsilon_n \coloneqq \max_{k\in[\![0,n-1]\!]} (x_{k+1} - x_k) \underset{n\to 0}{\longrightarrow} 0$$

we benefit from the regularity of $f$ and obtain a limit $I$ for $(I_n)_{n\in\mathbb{N}}$ while $f_n$ converges to $f$, at least pointwisely. All that is left to do is to call $I$ the integral of $f$:

$$\int_a^b f(x)dx \coloneqq I$$

Any serious development of Riemann integral now needs to check that the value obtained for the integral of $f$ is independent of the choice we made for the partitions, but this method is not the object of study here.

## Length and integrals

As long as we are measuring objects on the real line, the meaning of 'size' is length. Lengths and integrals are connected, as it is possible to construct one from the other.

It is important to note that in the above construction we went from length to integrals. Indeed, the length appeared when we integrated piecewise functions:

$$I_n = \sum_{k=0}^{n-1} f(x_k)\text{Length}([x_k, x_k + 1])$$

if we denote, for any real interval $[\alpha, \beta]$,

$$\text{Length}([\alpha, \beta]) \coloneqq \beta - \alpha$$

By reversing this relation, we can extend the definition of length to sets $M \subset \mathbb{R}$ for which their indicator function

$$\mathbb{1}_M(x) \coloneqq \begin{cases} 1 \text{ if } x \in M \\ 0 \text{ else} \end{cases}$$

is Riemann integrable through

$$\text{Length(M)} \coloneqq \int_{\mathbb{R}} \mathbb{1}_M(x)dx \tag{.1}$$

since

$$\int_{\mathbb{R}} \mathbb{1}_{[\alpha,\beta]}(x)dx = \int_\alpha^\beta 1dx = \beta - \alpha$$

Using this definition, it is possible to make less intuitive measurements, such as the length of the Cantor set.

## The Cantor set

Start from the unit interval

$$\mathbb{C}_0 := [0, 1]$$

and then iteratively remove a third of it in the middle of all its parts:

$$\forall n \in \mathbb{N}, \mathbb{C}_{n+1} := \frac{\mathbb{C}_n \cup (2 + \mathbb{C}_n)}{3} \quad (.2)$$

The limit object

$$\mathbb{C} := \bigcap_{n\in\mathbb{N}} \mathbb{C}_n$$

is called the Cantor set.

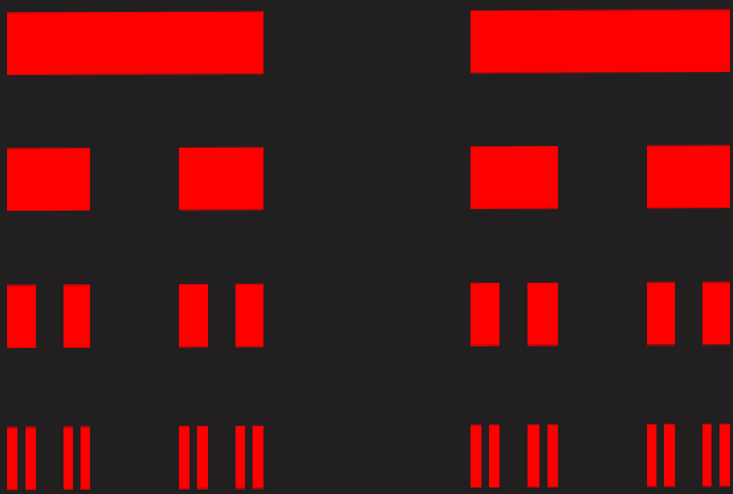


Figure 1: The 4 first steps in the construction of the Cantor set

The obvious question is then: what is the Length of $\mathbb{C}$? Easy enough: we started with a length of one and removed a third of it at each construction step, thus

$$\text{Length}(\mathbb{C}_n) = \left(\frac{2}{3}\right)^n$$

To prove this, we could evaluate the Length function in (.2) or notice that $\mathbb{C}_n$ is made up of $2^n$ intervals of length $3^{-n}$. Taking the piecewise function $\mathbb{1}_{\mathbb{C}_n}$ as Riemann approximation for $\mathbb{1}_{\mathbb{C}}$, it follows that

$$\text{Length}(\mathbb{C}) = \int_{\mathbb{R}} \mathbb{1}_{\mathbb{C}}(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}} \mathbb{1}_{\mathbb{C}_n}(x)dx = \lim_{n\to\infty} \text{Length}(\mathbb{C}_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

Hence, while remaining very much non-empty, the Cantor set has 0 length.

## The need for generalization

Proud of our success in measuring the Cantor set, we ask a more challenging question: what is the length of rational numbers, say for sake of boundedness, inside $[0, 1]$? Now, the non-Riemann integrability of $\mathbb{1}_{\mathbb{Q}\cap[0,1]}$ deprives us from an answer. Hoping that the reader is motivated by this existential problem, we will prove that the Lebesgue measure is a more precise measuring tool than the Length we just defined using Riemann integral. This will allow us to settle the question. In other words, the Lebesgue Integral will give a meaning to the following integral

$$\int_0^1 \mathbb{1}_{\mathbb{Q}}(x)dx$$

## Distributions

Imagine we have a point particle of mass 1 at the origin of space and nothing else. Then the density $\delta$ is 0 everywhere except at origin and must integrate to 1:

$$'\int_{\mathbb{R}^3} \delta(x)dx = 1'$$

This prevents the value of $\delta$ at the origin to be finite otherwise the above integral would be 0. Thus, $\delta$ cannot be defined as a real function, it is called a distribution. Let us think of the word distribution as in distribution probability (this is the case here since the integral is 1). Moreover, $\delta$ represents the physical distribution of mass in space.

Nevertheless, if we believe $\delta$ is a real function (which it is not) then $\forall M \subset \mathbb{R}^3$,

$$'\mathbb{1}_M(x)\delta(x) = \mathbb{1}_M(0_{\mathbb{R}^3})\delta(x)'$$

since $\forall x \neq 0, \delta(x) = 0$, therefore by linearity

$$'\int_{\mathbb{R}^3} \mathbb{1}_M(x)\delta(x)dx = \int_{\mathbb{R}^3} \mathbb{1}_M(0_{\mathbb{R}^3})\delta(x)dx = \mathbb{1}_M(0_{\mathbb{R}^3})\int_{\mathbb{R}^3} \delta(x)dx = \mathbb{1}_M(0_{\mathbb{R}^3})' \tag{.3}$$

Rigorously, $\delta$ can be defined as a measure called the Dirac mass. In this example, it is a measure on $\mathbb{R}^3$, meaning that the Dirac mass attributes a 'size' to objects in space, or in mathematical terms, to subsets of $\mathbb{R}^3$. Let $M \subset \mathbb{R}^3$, the precise definition is

$$\delta(M) \coloneqq \mathbb{1}_M(0_{\mathbb{R}^3})$$

Understand: 'the size of $M$ is one if $M$ contains the origin and 0 otherwise'. $\delta$ associate to $M$ the total mass that is found inside the region of space $M$. As promised, from this notion of 'size' follows a theory of integration. We rewrite (.3) as our new definition of the integral:

$$\int_{\mathbb{R}^3} \mathbb{1}_M(x)d\delta(x) \coloneqq \delta(M) \tag{.4}$$

In the right-hand integral we inserted the $\delta$ inside the usual $'dx'$ to remember that this integration is performed with respect to the Dirac measure as a tool for measuring sizes.

## Measures

Following the notation from the previous paragraph, we should replace the $'dx'$ in (.1) by $d\text{Length}(x)$ since the Riemann integral is constructed using the Length as measuring tool:

$$\int_{\mathbb{R}} \mathbb{1}_M(x)d\text{Length}(x) \coloneqq \text{Length(M)}$$

This is analogous to (.4), but with Length instead of $\delta$. In general, if we have a set $E$ and a way of measuring subsets of $E$, say, a function $\mu : \mathcal{P}(E) \to \mathbb{R}$, we will define an associated integral by

$$\forall M \subset E, \int_E \mathbb{1}_M(x)d\mu(x) \coloneqq \mu(M)$$

Once it is possible to integrate indicator functions we extend the integral to piecewise functions by linearity (a piecewise function is a linear combination of indicator functions) and then to a more general class of functions by taking limits, exactly as is done in Riemann integration. Making these steps rigorous will be the focus of a significant part of this course.

## Brief plan

The first part of these lecture notes will be devoted to methods for constructing interesting measures, with a focus on the Lebesgue measure generalizing the concept of length. From measures will follow the Lebesgue integral. We will study its basic properties and applications, ultimately leading us to an introduction to Fourier analysis.

# I Sigma-Algebras

Let $E$ be a set.

**Definition I.1:** *$\sigma$-algebra*

$\mathcal{A} \subseteq \mathcal{P}(E)$ is a $\sigma$-algebra on $E$ if it is

- non-empty: $\mathcal{A} \neq \varnothing$
- closed under complement: $\forall M \in \mathcal{A}, M^c \in \mathcal{A}$
- closed under countable unions: $\forall (M_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}, \bigcup_{n\in\mathbb{N}} M_n \in \mathcal{A}$.

$(E, \mathcal{A})$ is then called a measurable space and elements of $\mathcal{A}$ are called measurable sets.

**Example I.2**

- $\{\varnothing, E\}, \mathcal{P}(E)$ are respectively the smallest and the largest (for the inclusion) $\sigma$-algebras on $E$.
- $\{\varnothing, \{0,1\}, \{0\}, \{1,2\}\}$ is $\sigma$-algebra on $\{0,1,2\}$.

Beware that a $\sigma$-algebra on $E$ is not a subset of $E$, but a collection of subsets of $E$ as it is a subset of the powerset of $E$. When talking about a $\sigma$-Algebra $\mathcal{A}$ on $E$, the following assertions are true $\mathcal{A} \subseteq \mathcal{P}(E), E \in \mathcal{A}, M \in \mathcal{A} \implies M \subseteq E$ but $\mathcal{A} \in E$ is a mathematical statement that should not, under any circumstances, find its way into a student's exam.

Here are some immediate results

- The non-emptiness axiom can be replaced by $\varnothing \in \mathcal{A}$ or $E \in \mathcal{A}$. Indeed, if one of the previous is true then $\mathcal{A}$ is non-empty. Conversely, say that $M \in \mathcal{A}$, then by closeness under complement $M^c \in \mathcal{A}$, and by closeness under finite (a finite set is countable) union, $E = M \cup M^c \in \mathcal{A}$. Thus by closeness under complement again, $\varnothing = E^c \in \mathcal{A}$.
- With Morgan's law, a $\sigma$-algebra is also closed under countable intersections.
- Stability under difference: $M, N \in \mathcal{A} \implies M \backslash N = M \cup N^c \in \mathcal{A}$.

**Definition I.3:** *Generated $\sigma$-algebra*

Let $\mathcal{M} \subseteq \mathcal{P}(E)$,

$$\sigma(\mathcal{M}) := \bigcap_{\substack{\mathcal{A}\ \sigma-\text{algebra on } E \\ \mathcal{M}\subseteq\mathcal{A}}} \mathcal{A}$$

is the smallest $\sigma$-algebra containing $\mathcal{M}$. $\sigma(\mathcal{M})$ is called the $\sigma$-algebra generated by $\mathcal{M}$.

**Proof:**

An intersection of $\sigma$-algebras is a $\sigma$-algebra (use $\varnothing \in \mathcal{A}$ as the non-emptiness axiom to prove this as an Exercice), so $\sigma(\mathcal{M})$ is a $\sigma$-algebra and by construction, $\mathcal{M} \subseteq \sigma(\mathcal{M})$. Then, if $\mathcal{A}$ is a $\sigma$-algebra on $E$ containing $\mathcal{M}$, we have $\sigma(\mathcal{M}) \subseteq \mathcal{A}$.

We will be using this one last argument very often. For the purpose of rephrasing: $\sigma(\mathcal{M})$ is the unique $\sigma$-algebra on $E$ smaller than every other $\sigma$-algebra on $E$ containing $\mathcal{M}$.

**Example I.4**

On $\{0,1\}, \sigma(\{0\}) = \sigma(\{1\}) = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$.

**Definition I.5:** *Induced $\sigma$-algebra*

Let $F \subseteq E$ and $\mathcal{A}$ be a $\sigma$-algebra on $E$. Then $\mathcal{A}_F \coloneqq \{M \cap F, M \in \mathcal{A}\}$ is a $\sigma$-algebra on $F$ called the induced $\sigma$-algebra.

**Proof:**

**Step 1: Non-emptiness**

$\mathcal{A}_F$ is non-empty since it is the case for $\mathcal{A}$.

**Step 2: Closeness under complement**

Let $M \cap F \in \mathcal{A}_F$, then

$$F \backslash (M \cap F) = F \backslash M = \underbrace{M^c}_{\in \mathcal{A}} \cap F \in \mathcal{A}_F$$

(the complements are here understood inside of $E$).

**Step 3: Closeness under countable union**

Let $(M_n \cap F)_{n \in \mathbb{N}} \subseteq \mathcal{A}_F$,

$$\bigcup_{n \in \mathbb{N}} (M_n \cap F) = \underbrace{\left( \bigcup_{n \in \mathbb{N}} M_n \right)}_{\in \mathcal{A}} \cap F \in \mathcal{A}_F$$

**Remark I.6:** *Relation between generated and induced $\sigma$-algebras*

We ask about the relation between $\mathcal{A}_F$ and $\sigma\left(\{M \in \mathcal{A} | M \subseteq F\}\right)$ (generated inside of $F$):

- $\mathcal{A}_F$ is a $\sigma$-algebra on $F$ and $\{M \in \mathcal{A} | M \subseteq F\} \subseteq \mathcal{A}_F$ so $\sigma\left(\{M \in \mathcal{A} | M \subseteq F\}\right) \subseteq \mathcal{A}_F$.
- If $F \in \mathcal{A}$ then $\forall M \in \mathcal{A}, M \cap F \in \mathcal{A}$ and $M \cap F \subseteq F$ so

$$\mathcal{A}_F \subseteq \{M \in \mathcal{A} | M \subseteq F\} \subseteq \sigma\left(\{M \in \mathcal{A} | M \subseteq F\}\right)$$

  and thus $\mathcal{A}_F = \sigma\left(\{M \in \mathcal{A} | M \subseteq F\}\right)$

If $F \neq \mathcal{A}$ we can have equality or not

- Let $E \coloneqq \{0,1\}, \mathcal{A} \coloneqq \{\varnothing, \{0,1\}\}, F \coloneqq \{0\}$, we have

$$\begin{aligned}\mathcal{A}_F &= \{\varnothing, \{0\}\} \\ \sigma(\{M \in \mathcal{A} | M \subseteq F\}) &= \sigma(\{\varnothing\}) = \{\varnothing, \{0\}\}\end{aligned}$$

  so $\mathcal{A}_F = \sigma(\{M \in \mathcal{A} | M \subseteq F\})$.

- Let $E \coloneqq \{0,1,2,3\}, \mathcal{A} \coloneqq \{\varnothing, \{0,1\}, \{2,3\}, \{0,1,2,3\}\}, F \coloneqq \{0,2\}$, we have

$$\begin{aligned}\mathcal{A}_F &= \{\varnothing, \{0\}, \{2\}, \{0,2\}\} \\ \sigma(\{M \in \mathcal{A} | M \subseteq F\}) &= \sigma(\{\varnothing\}) = \{\varnothing, \{0,2\}\}\end{aligned}$$

  so $\sigma(\{M \in \mathcal{A} | M \subseteq F\}) \subset \mathcal{A}_F$.

**Definition I.7:** *Borel sets*

Let $\mathcal{T} \coloneqq \{\mathcal{O} \subseteq \mathbb{R} | \mathcal{O} \text{ is open}\}$, $\mathcal{B}(\mathbb{R}) \coloneqq \sigma(\mathcal{T})$ is called the Borel $\sigma$-algebra on $\mathbb{R}$ and its elements are called Borel sets.

The Borel $\sigma$-algebra is huge as it has to contain

- Open and closed sets
- Countable intersection of open sets, countable unions of closed sets
- Countable unions of countable intersection of open sets, countable intersection of countable unions of closed sets
- ...

**Definition I.8:** *Dynkin system ($\lambda$-system)*

$\mathcal{D} \subseteq \mathcal{P}(E)$ is a Dynkin system if

$$\begin{aligned}&\bullet\ E \in \mathcal{D} \\ &\bullet\ \forall N, M \subseteq \mathcal{D}, M \subseteq N \implies N \backslash M \in \mathcal{D} \\ &\bullet\ \forall (M_n)_{n \in \mathbb{N}} \subseteq \mathcal{D} \nearrow, \bigcup_{n \in N} M_n \in \mathcal{D}\end{aligned}$$

Let $\mathcal{M} \subseteq \mathcal{P}(E)$,

$$\lambda(\mathcal{M}) \coloneqq \bigcap_{\substack{\mathcal{D} \text{ Dynkin system on } E \\ \mathcal{M} \subseteq \mathcal{D}}} \mathcal{D}$$

is the smallest Dynkin system containing $\mathcal{M}$. $\lambda(\mathcal{M})$ is called the Dynkin system generated by $\mathcal{M}$.

Some facts:

- $\sigma$-Algebras are Dynkin systems.
- A Dynkin system is closed under complement and non-increasing intersections.
- $\{\varnothing, \{0,1\}, \{2,3\}, \{1,2\}, \{0,3\}, \{0,1,2,3\}\}$ is a Dynkin system in $\{0,1,2,3\}$ that is not a $\sigma$-algebra. Indeed it is not closed under union as $\{0,1\} \cup \{0,2\} = \{0,1,2\} \notin \mathcal{D}$
- On a finite set $E$ the closeness under increasing countable unions is always satisfied.

The advantage of Dynkin systems is that they are easier to construct than $\sigma$-algebras and can be used to obtain $\sigma$-algebras with the following result.

**Proposition I.9**

A Dynkin system which is closed under intersection if a $\sigma$-algebra.

**Proof:**

Let $\mathcal{D} \subseteq \mathcal{P}(E)$ be a Dynkin system closed under intersection. $\mathcal{D}$ is non-empty and closed under complement so it is also closed under finite unions.
Let $(M_n)_{n \in N} \subseteq \mathcal{D}$ and define

$$\widetilde{M}_n \coloneqq \bigcup_{i=0}^{n} M_i \in \mathcal{D}$$

then, $\left(\widetilde{M}_n\right)_{n \in N}$ is non-decreasing so

$$\bigcup_{n \in \mathbb{N}} \widetilde{M}_n = \bigcup_{n \in \mathbb{N}} M_n \in \mathcal{D}$$

**Theorem I.10:** *Dynkin*

Let $\mathcal{M} \subseteq \mathcal{P}(E)$ be non-empty and stable under intersection ($\mathcal{M}$ is called a $\pi$-system), then

- $\lambda(\mathcal{M}) = \sigma(\mathcal{M})$.
- if $\mathcal{D}$ is a Dynkin system such that $\mathcal{M} \subseteq \mathcal{D}$ then $\sigma(\mathcal{M}) \subseteq \mathcal{D}$.

For the moment, it is hard to see the usefulness of such an abstract theorem, but this will later lead to uniqueness of some measures. From this follows a Dynkin theorem about measures stating that if two measures coincide on $\mathcal{M}$ then they do so on $\lambda(\mathcal{M}) = \sigma(\mathcal{M})$. For example this can be applied to intervals if one let $\mathcal{M} \coloneqq \{]a,b[, a,b \in \mathbb{R}\}$, then $\sigma(\mathcal{M}) = \mathcal{B}(\mathbb{R})$. The conclusion is that only one measure on $\mathcal{B}(\mathbb{R})$ can extend the length of open intervals.

**Proof:**

$\sigma(\mathcal{M})$ is a Dynkin system and $\mathcal{M} \subseteq \sigma(\mathcal{M})$ hence $\lambda(\mathcal{M}) \subseteq \sigma(\mathcal{M})$.
For the converse inclusion we prove that $\lambda(\mathcal{M})$ is closed under intersection so $\lambda(\mathcal{M})$ will be $\sigma$-algebra containing $\mathcal{M}$. To move toward that goal, we prove that given $A \in \lambda(\mathcal{M})$,

$$\mu(A) \coloneqq \{B \in \lambda(\mathcal{M}) | A \cap B \in \lambda(\mathcal{M})\}$$

is a Dynkin system:

- $E \in \lambda(\mathcal{M})$ and $A \cap E = A \in \lambda(\mathcal{M})$ so $E \in \mu(A)$
- Let $B_1, B_2 \in \mu(A)$ such that $B_1 \subseteq B_2$, we have $B_2 \backslash B_1 \in \lambda(\mathcal{M})$ and

$$A \cap (B_2 \backslash B_1) = \underbrace{(A \cap B_2)}_{\in \lambda(\mathcal{M})} \backslash \underbrace{A \cap B_1}_{\in \lambda(\mathcal{M})} \in \lambda(\mathcal{M})$$

  since $A \cap B_1 \subseteq A \cap B_2$. So $B_2 \backslash B_1 \in \mu(A)$.

- Let $(B_n)_{n\in\mathbb{N}} \subseteq \mu(A) \nearrow$, then

$$A \cap \bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} \underbrace{(A \cap B_n)}_{\in \lambda(\mathcal{M}), \nearrow} \in \lambda(\mathcal{M})$$

  so $\bigcup_{n\in\mathbb{N}} B_n \in \mu(A)$.

Let $A \in \lambda(\mathcal{M})$, our goal is to prove that $\mu(A) = \lambda(\mathcal{M})$, i.e. that $\lambda(\mathcal{M})$ is closed under intersection.
First, assume that $B \in \mathcal{M}$, then $\mathcal{M} \subseteq \mu(B)$ as $\mathcal{M}$ is closed under intersection. Since $\mu(B)$ is a Dynkin system, $\lambda(\mathbf{M}) \subseteq \mu(B)$ and by construction $\lambda(\mathcal{M}) = \mu(B)$. We deduce that $A \in \mu(B)$ meaning that $A \cap B \in \lambda(\mathcal{M})$ thus $B \in \mu(A)$. We proved that $\mathcal{M} \subseteq \mu(A)$ so it follows that $\mu(A) = \lambda(\mathcal{M})$.
The second point is a direct consequence of the first as

$$\sigma(\mathcal{M}) = \lambda(\mathcal{M}) \subseteq \lambda(\mathcal{D}) = \mathcal{D}$$

# II Constructing measures

Let $(E, \mathcal{A})$ be a measurable space.

**Definition II.1:** *Measure*

$\mu : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ is called a measure if

- $\mu(\varnothing) = 0$
- $\forall \left(M_n\right)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ disjoints, $\mu\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \sum_{n\in\mathbb{N}} \mu(M_n)$ ($\sigma$-additivity)

$(E, \mathcal{A}, \mu)$ is then called a measured space.

**Example II.2**

- The counting measure on $(E, \mathcal{P}(E))$ defined $\forall M \in \mathcal{P}(E)$ by

$$\#(M) \coloneqq \begin{cases} \text{card(M) if } M \text{ is finite} \\ +\infty \text{ else} \end{cases}$$

- The dirac mass at $a \in E$ defined $\forall M \in \mathcal{P}(E)$ by

$$\delta_a(M) \coloneqq \mathbb{1}_M(a)$$

Let $(E, \mathcal{A}, \mu)$ be a measured space.

**Definition II.3:** *Vocabulary*

$\mu$ is said to be

- finite if $\mu(E) < +\infty$
- $\sigma$-finite if $\exists \left(M_n\right)_{n\in M} \subseteq \mathcal{A} | E = \bigcup_{n\in N} M_n$ and $\forall n \in \mathbb{N}, \mu(M_n) < +\infty$
- a probability if $\mu(E) = 1$

If $E$ is a topological space and $\mathcal{A} = \mathcal{B}(E)$ then $\mu$ is called a Borel measure. It is also called locally finite if it is finite on compacts sets. Additionally, we define its support to be

$$\text{supp}(\mu) \coloneqq \bigcap_{F \text{ closed} | \mu(F^c)=0} F$$

The support of a Borel measure is the smallest closed set containing all the mass (of 0 measure complement).

On a finite set $\#$ is finite, on a countable set it is $\sigma$-finite, on a non countable set it is neither of both.

$\delta_a$ is a probability measure and $\operatorname{supp}(\delta_a) = \{a\}$.

**Proposition II.4:** *Basic properties of measures*

1. $\mu$ is non-decreasing: if $M, N \in \mathcal{A} | M \subseteq N$ then $\mu(M) \leqslant \mu(N)$.
   Moreover if $\mu(M) < +\infty, \mu(N \backslash M) = \mu(N) - \mu(M)$

Let $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then

2. monotonous limit and measure exchange:

$$(M_n)_{n\in\mathbb{N}} \nearrow \implies \mu\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \lim_{n\in\mathbb{N}} \mu(M_n)$$
$$(M_n)_{n\in\mathbb{N}} \searrow \implies \mu\left(\bigcap_{n\in\mathbb{N}} M_n\right) = \lim_{n\in\mathbb{N}} \mu(M_n)$$

3. $\sigma$-sub-additivity:

$$\mu\left(\bigcup_{n\in\mathbb{N}} M_n\right) \leqslant \sum_{n\in\mathbb{N}} \mu(M_n)$$

**Proof:**

1. Since $N = M \sqcup (N \backslash M)$,
$$\mu(N) = \mu(M) + \mu(N \backslash M) \geqslant \mu(M)$$
   Moreover if $\mu(M) < +\infty, \mu(N \backslash M) = \mu(N) - \mu(M)$.

2. For $(M_n)_{n\in\mathbb{N}} \searrow$, let $\widetilde{M}_0 := M_0$ and $\forall n \in \mathbb{N}, \widetilde{M}_{n+1} := M_{n+1} \backslash M_n$. Let $\forall n \in \mathbb{N}, p \in \mathbb{N}^*$, since $\widetilde{M}_n \subseteq M_n$ and $M_n \subseteq M_{n+p-1}$,
$$\widetilde{M}_{n+p} \cap \widetilde{M}_n \subset (M_{n+p} \backslash M_{n+p-1}) \cap M_n = M_{n+p} \cap (M_n \backslash M_{n+p-1}) = M_{n+p} \cap \varnothing = \varnothing$$
   so $(\widetilde{M}_n)_{n\in\mathbb{N}}$ are disjoints. Then
$$\bigcup_{i=0}^{p} \widetilde{M}_i = M_0 \cap (M_1 \backslash M_0) \cup \cdots \cup (M_p \backslash M_{p-1}) = \bigcup_{i=0}^{p} M_i = M_p$$
$$\bigcup_{i\in\mathbb{N}} \widetilde{M}_i = \bigcup_{i\in\mathbb{N}} M_i$$
   so
$$\mu(M_p) = \sum_{i=0}^{p} \mu\left(\widetilde{M}_i\right)$$
   and
$$\mu\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} \widetilde{M}_n\right) = \sum_{n\in\mathbb{N}} \mu(\widetilde{M}_n) = \lim_{n\to\infty} \sum_{i=0}^{n} \mu(\widetilde{M}_i) = \lim_{n\to\infty} \mu(M_n)$$

For $(M_n)_{n\in\mathbb{N}} \searrow$, if the intersection has $+\infty$ as measure the equality is trivial since $\forall n \in \mathbb{N}, \mu(M_n) = +\infty$. Otherwise we pass to the complement: $\forall n \in \mathbb{N}, \widetilde{M_n} \coloneqq M_0 \backslash M_n$ so $(\widetilde{M_n})_{n\in\mathbb{N}} \nearrow$and

$$\mu\left(\bigcup_{n\in\mathbb{N}} \widetilde{M_n}\right) = \mu\left(M_0 \backslash \bigcap_{n\in\mathbb{N}} M_n\right) = -\mu\left(\bigcap_{n\in\mathbb{N}} M_n\right)$$
$$= \lim_{n\to\infty} (\widetilde{M_n}) = \mu(M_0) - \lim_{n\to\infty} \mu(M_n)$$

hence

$$\mu\left(\bigcap_{n\in\mathbb{N}} M_n\right) = \lim_{n\to\infty} \mu(M_n)$$

3. Starting with $M, N \in \mathcal{A}$, since $N \backslash M \subseteq N$

$$\mu(M \cup N) = \mu(N) + \mu(N\backslash M) \leqslant \mu(N) + \mu(M)$$

By induction this extend to finite families, and by the previous point

$$\mu\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=0}^{n} M_i\right) \leqslant \lim_{n\to\infty} \sum_{i=0}^{n} \mu(M_i) = \sum_{n\in\mathbb{N}} \mu(M_n)$$

**Theorem II.5:** *Dynkin theorem for measures*

Let $M \subseteq \mathcal{P}(E)$ be closed under intersection and such that $E \in \mathcal{M}$. Let $\mu, \nu$ be two finite measures on $(E, \sigma(\mathcal{M}))$ such that $\mu_{|\mathcal{M}} = \nu_{|\mathcal{M}}$, then $\mu = \nu$.

**Proof:**

We prove that $\mathcal{D} \coloneqq \{M \in \sigma(\mathcal{M}) | \mu(M) = \nu(M)\}$ is a Dynkin system:

- $E \in \mathcal{M} \subseteq \mathcal{D}$
- $M, N \in \mathcal{D} | M \subseteq N \implies \mu(N\backslash M) = \mu(N) - \mu(M) = \nu(N) - \nu(M) = \nu(N\backslash M)$
- $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{D} \nearrow \implies \mu\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \lim_{n\to\infty} \mu(M_n) = \lim_{n\to\infty} \nu(M_n) = \nu\left(\bigcup_{n\in\mathbb{N}} M_n\right)$

so by the Dynkin theorem for $\sigma$-algebras, since we have that $\mathcal{M} \subseteq \mathcal{D}$, we conclude that $\sigma(\mathcal{M}) \subseteq \mathcal{D}$ hence $\mu = \nu$.

Note that the last point of the proof that $\mathcal{D}$ is a Dynkin system really relies on the fact that the sequence $(M_n)_{n\in\mathbb{N}}$ is non-decreasing to be able to exchange the limits and the measures. We cannot use this proof on $\sigma$-algebra directly hence the need to introduce Dynkin systems.

**Definition II.6:** *induced measure*

Let $F \in \mathcal{A}$ then $\mu_F \coloneqq \mu_{|\mathcal{A}_F}$ is a measure on $(F, \mathcal{A}_F)$ called the induced measure on $F$. If $\mu(F) < +\infty$ then $\mu_F$ is the only measure on $(F, \mathcal{A}_F)$ that coincide with $\mu$ on $\{M \in \mathcal{A} | M \subseteq F\}$.

**Proof:**

Since $F \in \mathcal{A}$, we have that $\mathcal{A}_F \subseteq \mathcal{A}$ and $\mu_F$ is a measure on $(F, \mathcal{A}_F)$. The uniqueness of $\mu_F$ is a consequence of Remark I.6 which implies that

$$\mathcal{A}_F = \sigma\left(\{M \in \mathcal{A} | M \subseteq F\}\right)$$

and of Theorem II.5.

**Definition II.7:** *limits of sets*

Let $(M_n)_{n\in\mathbb{N}} \subseteq \mathbf{P}(E)$, we define

$$\liminf_{n\to\infty} M_n \coloneqq \bigcup_{n\in\mathbb{N}} \bigcap_{i=n}^{\infty} M_i$$

$$\limsup_{n\to\infty} M_n \coloneqq \bigcap_{n\in\mathbb{N}} \bigcup_{i=n}^{\infty} M_i$$

If these two sets are equal it is then called the limit of $(M_n)_{n\in\mathbb{N}}$.

Some basics facts:

- $\liminf_{n\to\infty} M_n \subseteq \limsup_{n\to\infty} M_n$
- $(M_n)_{n\to\infty} \nearrow \implies \liminf_{n\to\infty} M_n = \bigcup_{n\to\infty} M_n$
- $(M_n)_{n\to\infty} \searrow \implies \limsup_{n\to\infty} M_n = \bigcap_{n\to\infty} M_n$

**Lemma II.8:** *Borel-Cantelli*

Let $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then

$$\sum_{n\in\mathbb{N}} \mu(M_n) < +\infty \implies \mu\left(\limsup_{n\to\infty} M_n\right) = 0$$

This is not to be confused with the basic fact about convergent series

$$\limsup_{n\to\infty} \mu(M_n) = \mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{i=n}^{\infty} M_i\right)$$

By monotonicity it also follows that

$$\mu\left(\liminf_{n\to\infty} M_n\right) = 0$$

**Proof:**

By monotonicity and $\sigma$-sub-additivity,

$$\mu\left(\limsup_{n\to\infty} M_n\right) = \mu\left(\bigcap_{n\in\mathbb{N}}\bigcup_{i=n}^{\infty} M_i\right) \leqslant \mu\left(\bigcup_{i=n}^{\infty} M_i\right) \leqslant \sum_{i=n}^{\infty}\mu(M_n) \underset{n\to\infty}{\longrightarrow} 0$$

**Definition II.9:** *Negligible sets*

- Negligible sets of $(E, \mathcal{A}, \mu)$ are elements of

$$\mathcal{N}_\mu := \{N \in \mathcal{P}(E) | \exists M \in \mathcal{A} | N \subseteq M \text{ and } \mu(M) = 0\}$$

- If $\exists N \in \mathcal{N}_\mu$ such that a proposition $\mathcal{P}_x$ is true $\forall x \in E\backslash N$ then $\mathcal{P}_x$ is said to be true $\mu - a.e.$ (almost everywhere).
- If $\mathcal{N}_\mu \subseteq \mathcal{A}$, then $(E, \mathcal{A}, \mu)$ is called complete.

**Example II.10**

The absolute value is almost everywhere continuous for every measure that evaluate $\{0\}$ to 0.

The problem of a measure space not being complete is that we miss on the opportunity to measure sets whose measure should obviously be 0 since they are included in a measurable set of 0 measure. But we have a general process to make all measure complete by extending them to a lager $\sigma$-algebra.

**Proposition II.11:** *Completion of measures*

The following is a $\sigma$-algebra called the completed $\sigma$-algebra:

$$\overline{\mathcal{A}}^\mu := \{M \in \mathcal{P}(E) | \exists N_1, N_2 \in \mathcal{A} | N_1 \subseteq M \subseteq N_2 \text{ and } \mu(N_2\backslash N_1) = 0\} \tag{II.1}$$

and

1. $\overline{\mathcal{A}}^\mu = \sigma(\mathcal{A} \cup \mathcal{N}_\mu)$
2. The following it a well defined measure on $(E, \overline{\mathcal{A}}^\mu)$ called the completed measure:

$$\overline{\mu} : \begin{array}{ccc} \overline{\mathcal{A}}^\mu & \to & \overline{\mathbb{R}}_+ \\ M & \mapsto & \mu(N_1) = \mu(N_2) \end{array}$$

   Moreover, $\overline{\mu}$ is the unique measure on $(E, \overline{\mathcal{A}}^\mu)$ extending $\mu$.
3. $(E, \overline{\mathcal{A}}^\mu, \overline{\mu})$ is complete.

**Proof:**

Try it yourself following Subsection VII.1.

1. **Step 1:** $\overline{\mathcal{A}}^\mu \subseteq \sigma(\mathcal{A} \cup \mathcal{N}_\mu)$

Let $M \in \overline{\mathcal{A}}^{\mu}$ and $N_1, N_2$ according to (II.1), then $M \backslash N_1 \subseteq N_2 \backslash N_1 \in \mathcal{A}$ so $M \backslash N_1 \in \mathcal{N}_\mu$ and

$$M = \underbrace{(M \backslash N_1)}_{\in \mathcal{N}_\mu} \cup \underbrace{N_1}_{\in \mathcal{A}} \in \sigma(\mathcal{A} \cup \mathcal{N}_\mu)$$

**Step 2: $\mathcal{A} \cup \mathcal{N}_\mu \subseteq \overline{\mathcal{A}}^{\mu}$**

Taking $N_1 = N_2 = M$ we see that $\mathcal{A} \subseteq \overline{\mathcal{A}}^{\mu}$.
Let $N \in \mathcal{N}_\mu$, then $\exists M \in \mathcal{A}$ such that $N \subseteq M$ and $\mu(M) = 0$, thus choosing $N_1 = \varnothing$ and $N_2 = M$ we get that $N \in \overline{\mathcal{A}}^{\mu}$.

**Step 3: $\overline{\mathcal{A}}^{\mu}$ is a $\sigma$-algebra**

- $\mathcal{A} \subseteq \overline{\mathcal{A}}^{\mu}$ so $\overline{\mathcal{A}}^{\mu} \neq \varnothing$
- Let $M \in \overline{\mathcal{A}}^{\mu}$ and $N_1, N_2$ according to (II.1), we prove that $M^c \in \overline{\mathcal{A}}^{\mu}$ by noticing that $N_2^c \subseteq M^c \subseteq N_1^c$ and $N_1^c, N_2^c \in \mathcal{A}$ and $N_1^c \backslash N_2^c = N_2 \backslash N_1$.
- Let $(M_n)_{n\in\mathbb{N}} \subseteq \overline{\mathcal{A}}^{\mu}$ and $(N_{1,n}, N_{2,n})_{n\in\mathbb{N}}$ according to (II.1), then

$$\underbrace{\bigcup_{n\in\mathbb{N}} N_{1,n}}_{\in \mathcal{A}} \subseteq \bigcup_{n\in\mathbb{N}} M_n \subseteq \underbrace{\bigcup_{n\in\mathbb{N}} N_{2,n}}_{\in \mathcal{A}}$$

  and

$$\left(\bigcup_{n\in\mathbb{N}} N_{2,n}\right) \backslash \bigcup_{n\in\mathbb{N}} N_{1,n} \subseteq \bigcup_{n\in\mathbb{N}} N_{2,n} \backslash N_{1,n}$$

  so by $\sigma$-sub-additivity,

$$\mu\left(\left(\bigcup_{n\in\mathbb{N}} N_{2,n}\right) \backslash \bigcup_{n\in\mathbb{N}} N_{1,n}\right) \leqslant \sum_{n\in\mathbb{N}} \mu(N_{2,n} \backslash N_{1,n}) = 0$$

  This proves that

$$\bigcup_{n\in\mathbb{N}} M_n \in \overline{\mathcal{A}}^{\mu}$$

**Step 4: Conclusion**

The two previous steps imply that $\mathcal{A} \cup \mathcal{N}_\mu \subseteq \overline{\mathcal{A}}^{\mu}$

2. **Step 5: $\overline{\mu}$ is well defined**

Let $M \in \overline{\mathcal{A}}^{\mu}$ and assume that $\exists N_1, N_2, \widetilde{N}_1, \widetilde{N}_2 \in \mathcal{A}$ such that

$$N_1 \subseteq M \subseteq N_2 \quad \widetilde{N}_1 \subseteq M \subseteq \widetilde{N}_2$$
$$\mu(N_2 \backslash N_1) = \mu\left(\widetilde{N}_2 \backslash \widetilde{N}_1\right) = 0$$

First remark that

$$\mu(N_2) = \mu(N_2 \backslash N_1) + \mu(N_1) = \mu(N_1)$$

and similarly $\mu(\widetilde{N}_2) = \mu\left(\widetilde{N}_1\right)$. Then, since

$$\underbrace{\left(N_2 \cap \widetilde{N}_2\right)}_{\in \mathcal{A}} \backslash N_1 \subseteq N_2 \backslash N_1$$

we get that $\left(N_2 \cap \widetilde{N}_2\right) \backslash N_1 = 0$ so

$$\mu\left(N_2 \cap \widetilde{N}_2\right) = \mu(N_1)$$

Exchanging the roles of $N_1, N_2$ and $\widetilde{N}_1, \widetilde{N}_2$, we obtain

$$\mu\left(N_2 \cap \widetilde{N}_2\right) = \mu\left(\widetilde{N}_1\right)$$

so $\mu(N_1) = \mu(N_2) = \mu\left(\widetilde{N}_1\right) = \mu\left(\widetilde{N}_2\right)$.

**Step 6: $\overline{\mu}$ extends $\mu$**

If $M \in \mathcal{A}$, taking $N_1 = N_2 = M$ we see that

$$\overline{\mu}(M) = \mu(N_1) = \mu(M)$$

**Step 7: $\overline{\mu}$ is a measure**

- Since $\varnothing \in \mathcal{A}$, $\overline{\mu}(\varnothing) = \mu(\varnothing) = 0$.
- Let $(M_n)_{n\in\mathbb{N}} \subseteq \overline{\mathcal{A}}^{\mu^*}$ disjoints and $(N_{1,n})_{n\in\mathbb{N}}$ according to (II.1) which are also disjoints since $\forall n \in \mathbb{N}, N_{1,n} \subseteq M_n$. Hence

$$\overline{\mu}\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} N_{1,n}\right) = \sum_{n\in\mathbb{N}} \mu\left(N_{1,n}\right) = \sum_{n\in\mathbb{N}} \overline{\mu}(M_n)$$

**Step 8: Uniqueness**

Assume that $\eta$ is another measure on $(E, \overline{\mathcal{A}}^{\mu})$ such that $\eta_{|\mathcal{A}} = \mu$. Let $M \in \overline{\mathcal{A}}^{\mu}$ and $N_1, N_2$ according to (II.1), then

$$\eta(M) = \eta(M \backslash N_1) + \eta(N_1)$$

But

$$\eta(\underbrace{N_1}_{\in \mathcal{A}}) = \mu(N_1) = \overline{\mu}(M) \text{ and } \eta(M \backslash N_1) \leqslant \eta(\underbrace{N_2 \backslash N_1}_{\in \mathcal{A}}) = \mu(N_2 \backslash N_1) = 0$$

so $\eta(M) = \overline{\mu}(M)$.

3. Let $M \in \overline{\mathcal{A}}^{\mu}$ and $N_1, N_2$ according to (II.1). Assume that $\overline{\mu}(M) = 0$ and let $N \subseteq M$. Then $N \subseteq N_2 \in \mathcal{A}$ and $\mu(N_2) = \overline{\mu}(M) = 0$ so $N \in \mathcal{N}_{\mu} \subseteq \overline{\mathcal{A}}^{\mu}$.

# III Lebesgue's measure

**Definition III.1:** *Outer measure*

$\mu^* : \mathcal{P}(E) \to \overline{\mathbb{R}}_+$ is an outer measure on $E$ if

- $\mu^*(\varnothing) = 0$
- $\mu^*$ is non decreasing: $\forall M, N \in \mathcal{P}(E), M \subseteq N \implies \mu^*(M) \leqslant \mu^*(N)$
- $\mu^*$ is $\sigma$-sub-additive: $\forall (M_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}(E), \mu^*\left(\bigcup_{n\in\mathbb{N}} M_n\right) \leqslant \sum_{n\in\mathbb{N}} \mu^*(M_n)$

In addition, $M \in \mathcal{P}(E)$ is called $\mu^*$-measurable if $\forall N \in \mathcal{P}(E)$,

$$\mu^*(N) = \mu^*(N \cap M) + \mu^*(N \backslash M)$$

and the set of $\mu^*$-measurable sets is denoted $\mathcal{A}_{\mu^*}$.

Remark that with the $\sigma$-sub-additivity, it is enough the check that

$$\forall N \in \mathcal{P}(E), \mu^*(N) \geqslant \mu^*(N \cap M) + \mu^*(N \backslash M)$$

to prove the $\mu^*$-measurability of $M$.

**Definition III.2:** *Lebesgue outer measure*

Let $M \subseteq \mathbb{R}$, define

$$\mathcal{I}(M) \coloneqq \left\{(a_n, b_n)_{n\in\mathbb{N}} \,|\forall n \in \mathbb{N}, a_n \leqslant b_n \text{ and } M \subseteq \bigcup_{n\in\mathbb{N}} ]a_n, b_n[\right\}$$
$$l^*(M) = \inf_{(a_n,b_n)_{n\in\mathbb{N}}\in\mathcal{I}(M)} \sum_{n\in\mathbb{N}} (b_n - a_n)$$

$\mathcal{I}(M)$ is the set of open intervals covering of $M$. Given such a covering $(a_n, b_n)_{n\in\mathbb{N}} \in \mathcal{I}(M)$, the value

$$\sum_{n\in\mathbb{N}} (b_n - a_n)$$

is an upper bound of what we would to define as the length of $M$. This taking the infimum over $\mathcal{I}(M)$ we attribute to $M$ the length of the smallest possible covering by open intervals (the minimizer might not exists).

**Proof:**

First, note that $\forall M \subseteq \mathbb{R}, \mathcal{I}(M) \neq \varnothing$ since $(-n, n)_{n\in\mathbb{N}} \in \mathcal{I}(M)$ as

$$M \subseteq \mathbb{R} = \bigcup_{n\in\mathbb{N}} ]-n, n[$$

So the Lebesgue outer measure is well defined as an infimum over positive values.

We check that $l^*$ is indeed an outer measure:

- $(0,0) \in \mathcal{I}(\varnothing)$ so $l^*(\varnothing) \leqslant 0 - 0 = 0$
- Let $M, N \in \mathcal{P}(\mathbb{R}) | M \subseteq N$, a covering of $N$ is a covering of $M$ so $\mathcal{I}(N) \subseteq \mathcal{I}(M)$ and as a consequence $l^*(M) \leqslant l^*(N)$.
- Let $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}(E)$, let $\epsilon > 0$, then by property of the infimum, $\forall n \in \mathbb{N}, \exists (a_{n,m}, b_{n,m})_{m\in\mathbb{N}} \in \mathcal{I}(M_n)$ such that

$$l^*(M_n) \geqslant \sum_{m\in\mathbb{N}} (b_{n,m} - a_{n,m}) - \frac{\epsilon}{2^n}$$

then

$$\bigcup_{n\in\mathbb{N}} M_n \subseteq \bigcup_{n,m\in\mathbb{N}} ]a_{n,m}, b_{n,m}[$$

which means that

$$(a_{n,m}, b_{n,m})_{n,m\in\mathbb{N}} \in \mathcal{I}\left(\bigcup_{n\in\mathbb{N}} M_n\right)$$

so

$$l^*\left(\bigcup_{n\in\mathbb{N}} M_n\right) \leqslant \sum_{n,m\in\mathbb{N}} (b_{n,m} - a_{n,m}) \leqslant \sum_{n\in\mathbb{N}} \left(l^*(M_n) + \frac{\epsilon}{2^n}\right) = \sum_{n\in\mathbb{N}} l^*(M_n) + 2\epsilon$$

and one gets $\sigma$-sub-additivity of $l^*$ by taking $\epsilon \to 0$.

**Proposition III.3**

1. $\forall a \leqslant b, l^*([a,b]) = b - a$

Let $M \in \mathcal{P}(\mathbb{R}), \lambda \in \mathbb{R}$ we denote $\lambda M \coloneqq \{\lambda x, x \in M\}$ and $\lambda + M \coloneqq \{\lambda + x, x \in M\}$.

2. $l^*$ is translation invariant: $l^*(\lambda + M) = l^*(M)$
3. $l^*$ scaled with dilatations: $l^*(\lambda M) = |\lambda|\, l^*(M)$

**Proof:**

1. Let $\epsilon > 0$, $]a-\epsilon, b+\epsilon[ \in \mathcal{I}([a,b])$ so $l^*([a,b]) \leqslant b-a+2\epsilon$. Taking $\epsilon \to 0$, $l^*([A,b]) \leqslant b-a$.
   Let $(a_n, b_n)_{n\in\mathbb{N}} \in \mathcal{I}([a,b])$, by compactness of $[a,b], \exists N \in \mathbb{N}$ such that

$$[a,b] \subseteq \bigcup_{n=0}^{N} ]a_n, b_n[$$

then we can prove by induction (Exercice) that

$$b - a \leqslant \sum_{n=0}^{N} (b_n - a_n)$$

so

$$\sum_{n\in\mathbb{N}}^{N} (b_n - a_n) \geqslant b - a$$

and passing to the infimum, $l^*([a,b]) \geqslant b - a$.

2. Note that

$$\begin{aligned}(a_n, b_n)_{n\in\mathbb{N}} \in \mathcal{I}(\lambda + M) &\iff \lambda + M \subseteq \bigcup_{n\in\mathbb{N}} ]a_n, b_n[ \iff M \subseteq \bigcup_{n\in\mathbb{N}} ]a_n - \lambda, b_n - \lambda[ \\ &\iff (a_n - \lambda, b_n - \lambda)_{n\in\mathbb{N}} \in \mathcal{I}(M)\end{aligned}$$

so

$$\begin{aligned}l^*(\lambda + M) &= \inf_{(a_n-\lambda, b_n-\lambda)_{n\in\mathbb{N}}\in\mathcal{I}(M)} \sum_{n\in\mathbb{N}} (b_n - a_n) = \inf_{(a_n, b_n)_{n\in\mathbb{N}}\in\mathcal{I}(M)} \sum_{n\in\mathbb{N}} (b_n - \lambda - a_n + \lambda) \\ &= l^*(M)\end{aligned}$$

3. Similar, see notes

**Proposition III.4:** *From outer measure to measure*

$\left(E, \mathcal{A}_{\mu^*}, \mu^*_{|\mathcal{A}_{\mu^*}}\right)$ is a complete measured space.

**Proof:**

Try it yourself following Subsection VII.2.

**Step 1: $\mathcal{A}_{\mu^*}$ is closed under intersection**

Let $A, B \in \mathcal{A}_{\mu^*}$ and $N \in \mathcal{P}(E)$. Decomposing $N$ with respect to $A$ and then $N \cap A$ and $N \cap A^c$ with respect to $B$ we obtain

$$\begin{aligned}\mu^*(N) &= \mu^*(N \cap A) + \mu^*(N \cap A^c) \\ &= \mu^*(N \cap A \cap B) + \mu^*(N \cap A \cap B^c) + \mu^*(N \cap A^c \cap B) + \mu^*(N \cap A^c \cap B^c)\end{aligned} \quad \text{(III.1)}$$

Applying this to $N \cap (A \cup B)$ instead of $N$ one gets

$$\mu^*\left(N \cap (A \cup B)\right) = \mu^*(N \cap A \cap B) + \mu^*(N \cap A \cap B^c) + \mu^*(N \cap A^c \cap B) \quad \text{(III.2)}$$

since

$$(A \cup B) \cap (A \cap B) = A \cap B$$

$$\begin{aligned}(A \cup B) \cap (A \cap B^c) &= A \cap B^c \\ (A \cup B) \cap (A^c \cap B) &= A^c \cap B \\ (A \cup B) \cap (A^c \cap B^c) &= \varnothing\end{aligned}$$

Inserting (III.2) in (III.1) we obtain

$$\mu^*(N) = \mu^*\left(N \cap (A \cup B)\right) + \mu^*(N \cap A^c \cap B^c)$$

meaning that $A \cup B \in \mathcal{A}_{\mu^*}$

**Step 2: Disjoint unions**

Assuming that $A \cap B = \varnothing$, since $A \subseteq B^c$ and $B \subseteq A^c$, (III.2) becomes

$$\mu^*\left(N \cap (A \cup B)\right) = \mu^*(N \cap A) + \mu^*(N \cap B)$$

By induction, this generalizes to: $\forall n \in \mathbb{N}, \forall (A_i)_{i=0:n} \subseteq \mathcal{A}_{\mu^*}$ disjoints,

$$\mu^*\left(N \cap \left(\bigcup_{i=0}^{n} A_i\right)\right) = \sum_{i=0}^{n} \mu^*(N \cap A_i) \tag{III.3}$$

**Step 3: $\mathcal{A}_{\mu^*}$ is a $\sigma$-algebra**

- Let $N \in \mathcal{P}(E)$,

  $$\mu^*(N) = \mu^*(N \cap \varnothing) + \mu^*(N \cap E)$$

  so $\varnothing \in \mathcal{A}_{\mu^*}$.
- By definition of $\mathcal{A}_{\mu^*}$, $M \in \mathcal{A}_{\mu^*} \iff M^c \in \mathcal{A}_{\mu^*}$ so $\mathcal{A}_{\mu^*}$ is closed under complement.
- Let $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_{\mu^*}$, define

  $$\begin{aligned}&\widetilde{M}_0 \coloneqq M_0 \\ &\forall n \in \mathbb{N}, \widetilde{M}_{n+1} \coloneqq M_{n+1} \backslash \bigcup_{i=0}^{n} M_i\end{aligned}$$

  so that $\forall n \in \mathbb{N}$,

  $$\bigcup_{i=0}^{n} M_i = \bigcup_{i=0}^{n} \widetilde{M}_i \in \mathcal{A}_{\mu^*}$$

  and $\left(\widetilde{M}_n\right)_{n\in\mathbb{N}}$ are not disjoints. Let $N \in \mathcal{P}(E)$, we can apply (III.3) and then use the monotonicity of $\mu^*$:

  $$\mu^*(N) = \mu^*\left(N \cap \bigcup_{i=0}^{n} \widetilde{M}_i\right) + \mu^*\left(N \cap \left(\bigcup_{i=0}^{n} \widetilde{M}_i\right)^c\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} \mu^*\left(N \cap \widetilde{M}_i\right) + \mu^*\left(N \cap \left(\bigcup_{i=0}^{n} M_i\right)^c\right) \\
&\geqslant \sum_{i=0}^{n} \mu^*\left(N \cap \widetilde{M}_i\right) + \mu^*\left(N \cap \left(\bigcup_{i\in\mathbb{N}} M_i\right)^c\right)
\end{aligned}
$$

Taking the limit $n \to \infty$ and using $\sigma$-sub-additivity,

$$
\begin{aligned}
\mu^*(N) &\geqslant \sum_{i\in\mathbb{N}} \mu^*\left(N \cap \widetilde{M}_i\right) + \mu^*\left(N \cap \left(\bigcup_{i\in\mathbb{N}} M_i\right)^c\right) \\
&\geqslant \mu^*\left(\bigcup_{i\in\mathbb{N}} (N \cap \widetilde{M}_i)\right) + \mu^*\left(N \cap \left(\bigcup_{i\in\mathbb{N}} M_i\right)^c\right) \\
&= \mu^*\left(N \cap \bigcup_{i\in\mathbb{N}} \widetilde{M}_i\right) + \mu^*\left(N \cap \left(\bigcup_{i\in\mathbb{N}} M_i\right)^c\right) \\
&= \mu^*\left(N \cap \bigcup_{i\in\mathbb{N}} M_i\right) + \mu^*\left(N \cap \left(\bigcup_{i\in\mathbb{N}} M_i\right)^c\right) \qquad \text{(III.4)}
\end{aligned}
$$

meaning that

$$
\bigcup_{i\in\mathbb{N}} M_i \in \mathcal{A}_{\mu^*}
$$

**Step 4: $\mu_{|\mathcal{A}_{\mu^*}}$ is a measure**

With the previous conclusion we now know that (III.4) is an equality so

$$
\mu^*(N) = \sum_{i\in\mathbb{N}} \mu^*\left(N \cap \widetilde{M}_i\right) + \mu^*\left(N \cap \left(\bigcup_{i\in\mathbb{N}} M_i\right)^c\right)
$$

Assuming that $(M_n)_{n\in\mathbb{N}}$ are disjoint, $\forall i \in \mathbb{N}, \widetilde{M}_i = M_i$, so by choosing

$$
N \coloneqq \bigcup_{n\in\mathbb{N}} M_n
$$

we get

$$
\begin{aligned}
\mu^*\left(\bigcup_{n\in\mathbb{N}} M_n\right) &= \sum_{i\in\mathbb{N}} \mu^*\left(\left(\bigcup_{n\in\mathbb{N}} M_n\right) \cap M_i\right) + \mu^*\left(\left(\bigcup_{n\in\mathbb{N}} M_n\right) \cap \left(\bigcup_{i\in\mathbb{N}} M_i\right)^c\right) \\
&= \sum_{n\in\mathbb{N}} \mu^*(M_n)
\end{aligned}
$$

$\mu^*(\varnothing) = 0$ so $\mu^*_{|\mathcal{A}_{\mu^*}}$ is a measure.

**Step 5: completeness**

Let $M \in \mathcal{A}_{\mu^*}$ with $\mu^*(M) = 0$ and $A \subseteq M$. Let $N \in \mathcal{P}(E)$, since $N \cap A \subseteq M$,

$$
\mu^*(N \cap A) + \mu^*(N \cap A^c) \leqslant \mu^*(M) + \mu^*(N) = \mu^*(N)
$$

so $A \in \mathcal{A}_{\mu^*}$.

**Definition III.5:** *Lebesgue sets*

The Lebesgue $\sigma$-algebra on $\mathbb{R}$ is $\mathcal{L}(\mathbb{R}) \coloneqq \mathcal{A}_{l*}$, its elements are called real Lebesgue sets and the Lebesgue measure on $\mathbb{R}$ is $l \coloneqq l^*_{|\mathcal{L}(\mathbb{R})}$

**Theorem III.6**

$\mathcal{L}(\mathbb{R}) = \overline{\mathcal{B}(\mathbb{R})}$

We have the following strict inclusions (see tutorials)

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$$

We notice that we provided two different constructions of the Lebesgue $\sigma$-algebra:

- $\overline{\mathcal{B}(\mathbb{R})}$ the completion of the Borel $\sigma$-algebra which is well suited for proving that sets are Lebesgue sets (for example all Borel sets are immediately Lebesgue sets).
- $\mathcal{A}_{l*}$ which is interesting to compute the Lebesgue measure of a given measurable sets using the explicit defintion of $l^*$. Since we proved it for the Lebesgue outer measure, we know that the Lebesgue measure evaluate each real interval to its length.

A closed interval $[a, b]$ is a Borel set thus it is a Lebesgue set, so using Proposition III.3,

$$l([a,b]) = l^*([a,b]) = b - a$$

so

$$l(]a,b]) = l([a,b]) - l(\{a\}) = l([a,b])$$

and similarly

$$l([a,b]) = l(]a,b]) = l([a,b[) = l(]a,b[)$$

Another consequence is that if $M \in \mathcal{L}(\mathbb{R})$ is countable,

$$l(M) = l\left(\bigcup_{x \in M} \{x\}\right) = \sum_{x \in M} l(\{x\}) = \sum_{x \in M} 0 = 0$$

Since $\mathbb{Q} \in \mathcal{L}(\mathbb{R})$ is a Borel set as countable union of closed sets, we obtain that $l(\mathbb{Q}) = 0$.

Finally, we remark that $l$ is $\sigma$-finite since

$$\forall n \in \mathbb{N}, l([-n,n]) = 2n \text{ and } \mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n,n]$$

**Proof:**

**Step 1: $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$**

Since $\mathcal{B}(\mathbb{R}) = \sigma(\{]-\infty, a[, a \in \mathbb{R}\})$ it is sufficient to prove that $\forall a \in \mathbb{R}, ]-\infty, a[\in \mathcal{L}(\mathbb{R})$. Let $a \in \mathbb{R}, N \in \mathcal{P}(E)$, our goal is to prove that

$$l^*(\,]-\infty, a[\,) = l^*(\,]-\infty, a[\,\cap N) + l^*(\,]-\infty, a[\,\setminus N)$$

Let $(a_n, b_n)_{n\in\mathbb{N}} \in \mathcal{I}(N), \epsilon > 0$, we have that

$$\begin{aligned}
]a_n, b_n[\,\cap\,]-\infty, a[\, &= \,]\min(a, a_n), \min(a, b_n)[\\
]a_n, b_n[\,\cap[a, +\infty[\, &\subseteq \,]\max(a, a_n) - \frac{\epsilon}{2^n}, \max(a, b_n)[
\end{aligned}$$

so

$$\begin{aligned}
]-\infty, a[\,\cap N &\subseteq \bigcup_{n\in\mathbb{N}} \,]\min(a, a_n), \min(a, b_n)[\\
]-\infty, a[\,\setminus N &\subseteq \bigcup_{n\in\mathbb{N}} \,]\max(a, a_n) - \frac{\epsilon}{2^n}, \max(a, b_n)[
\end{aligned}$$

and

$$\begin{aligned}
&l^*(\,]-\infty, a[\,\cap N) + l^*(\,]-\infty, a[\,\setminus N)\\
&\leqslant \sum_{n\in\mathbb{N}} \left(\min(a, b_n) - \min(a, a_n) + \max(a, b_n) - \max(a, a_n) + \frac{\epsilon}{2^n}\right)\\
&= \sum_{n\in\mathbb{N}} (a + b_n - a - a_n) + 2\epsilon \underset{\epsilon\to 0}{\longrightarrow} \sum_{n\in\mathbb{N}} (b_n - a_n)
\end{aligned}$$

Taking the infimum over $\mathcal{I}(N)$, we obtain that

$$l^*(\,]-\infty, a[\,\cap N) + l^*(\,]-\infty, a[\,\setminus N) \leqslant l^*(N)$$

**Step 2: $\overline{\mathcal{B}(\mathbb{R})} \subseteq \mathcal{L}(\mathbb{R})$**

Assume that $N \in \mathcal{P}(E)$ is a negligible set of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), l)$, then $\exists M \in \mathcal{B}(\mathbb{R})$ such that $N \subseteq M$ and $l(M) = 0$. But $\mathcal{L}(\mathbb{R})$ is complete and $M \in \mathcal{L}(\mathbb{R})$ so $N \in \mathcal{L}(\mathbb{R})$. Therefore

$$\mathcal{N}^{l_{|\mathcal{B}(\mathbb{R})}} \cup \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$$

Moreover $\mathcal{L}(\mathbb{R})$ is a $\sigma$-algebra so

$$\overline{\mathcal{B}(\mathbb{R})} = \sigma\left(\mathcal{N}^{l_{|\mathcal{B}(\mathbb{R})}} \cup \mathcal{B}(\mathbb{R})\right) \subseteq \mathcal{L}(\mathbb{R})$$

**Step 3: $\mathcal{L}(\mathbb{R}) \subseteq \overline{\mathcal{B}(\mathbb{R})}$**

Let $M \in \mathcal{L}(\mathbb{R})$ we construct a bigger Borel set of same measure.
$\forall k \in \mathbb{N}^*, \exists (a_{n,k}, b_{n,k})_{n\in\mathbb{N}} \in \mathcal{I}(M)$ such that

$$l^*(M) \geqslant \sum_{n\in\mathbb{N}} (b_{n,k} - a_{n,k}) - \frac{1}{k}$$

Then, let

$$B_M := \bigcap_{k\in\mathbb{N}^*} \bigcup_{n\in\mathbb{N}} ]a_{n,k}, b_{n,k}[ \in \mathcal{B}(\mathbb{R})$$

Then $\forall k \in \mathbb{N}^*$,

$$l^*(B_M) \leqslant l^*\left(\bigcup_{n\in\mathbb{N}} ]a_{n,k}, b_{n,k}[\right) \leqslant \sum_{n\in\mathbb{N}} (b_{n,k} - a_{n,k}) \leqslant l^*(M) + \frac{1}{k}$$

so $l^*(B_M) \leqslant l^*(M)$. Since $M \subseteq B_M$ the proved that $l^*(B_M) = l(B_M) = l^*(M) = l(M)$. Applying this to $M_n := [-n, n] \cap M, N_n := [-n, n] \backslash M_n \in \mathcal{L}(\mathbb{R})$ we get $B_{M_n}, B_{N_n} \in \mathcal{B}(\mathbb{R})$ such that

$$\begin{aligned} &M_n \subseteq B_{M_n} \\ &N_n \subseteq B_{N_n} \\ &l\left(B_{M_n}\right) = l\left(M_n\right) \\ &l\left(B_{N_n}\right) = l\left(N_n\right) = 2n - l(M_n) \end{aligned}$$

so $[-n, n] \backslash B_{N_n} \subseteq [-n, n] \backslash N_n = M_n \subseteq B_{M_n}$ and

$$l\left(B_{M_n} \backslash \left([-n, n] \backslash B_{N_n}\right)\right) = l\left(B_{M_n}\right) - \left(2n - l\left(B_{N_n}\right)\right) = l\left(B_{M_n}\right) - l\left(B_{M_n}\right) = 0$$

so $M_n \in \overline{\mathcal{B}(\mathbb{R})}$ and

$$M = \bigcup_{n\in\mathbb{N}} M_n \in \overline{\mathcal{B}(\mathbb{R})}$$

**Theorem III.7:** *Characterization of the Lebesgue measure*

The Lebesgue measure is the unique measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ extending the length of intervals.

**Proof:**

Let $\mu$ be another measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ such that $\forall a \leqslant b, \mu([a, b]) = b - a$. Then $\forall n \in \mathbb{N}, \mu([-n, n]) = l([-n, n]) = 2n$. By Theorem II.5,

$$\mu_{|\mathcal{B}([n,-n])} = l_{|\mathcal{B}([n,-n])}$$

Let $M \in \mathcal{B}(\mathbb{R})$ and denote $M_n := M \cap [-n, n] \in \mathcal{B}([-n, n])$ (exercice), then

$$l(M) = l\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \lim_{n\to\infty} l(M_n) = \lim_{n\to\infty} \mu(M_n) = \mu\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \mu(M)$$

So

$$\mu_{|\mathcal{B}(\mathbb{R})} = l_{|\mathcal{B}(\mathbb{R})}$$

and by uniqueness of the completed measure (Proposition II.11)

$$\mu_{|\overline{\mathcal{B}(\mathbb{R})}} = l_{|\overline{\mathcal{B}(\mathbb{R})}}$$

but since $\overline{\mathcal{B}(\mathbb{R})} = \mathcal{L}(\mathbb{R})$ (Theorem III.6) we conclude that $\mu = l$.

**Proposition III.8**

Let $M \in \mathcal{P}(\mathbb{R}), \lambda \in \mathbb{R}$

- $l$ is translation invariant: $l(\lambda + M) = l(M)$
- $l$ scaled with dilatations: $l(\lambda M) = |\lambda|\, l(M)$

**Proof:**

**Step 1: $\mathcal{L}(\mathbb{R})$ is closed under translation:**

Let $M \in \mathcal{L}(\mathbb{R}), N \in \mathcal{P}(\mathbb{R}), \lambda \in \mathbb{R}$,

$$\begin{aligned} l^*\left(N \cap (\lambda + M)\right) + l^*(N \backslash (\lambda + M)) &= l^*(\lambda + (N - \lambda) \cap M) + l^*(\lambda + (N - \lambda) \backslash M) \\ &= l^*((N - \lambda) \cap M) + l^*((N - \lambda) \backslash M) \\ &= l^*(N - \lambda) = l^*(N) \end{aligned}$$

so $\lambda + M \in \mathcal{L}(\mathbb{R})$.

**Step 2: $\mathcal{L}(\mathbb{R})$ is closed under dilatations:**

Let $M \in \mathcal{L}(\mathbb{R}), N \in \mathcal{P}(\mathbb{R}), \lambda \in \mathbb{R}$, if $\lambda = 0$, then $\lambda M = \{0\} \in \mathcal{L}(\mathbb{R})$. Otherwise,

$$\begin{aligned} l^*\left(N \cap (\lambda M)\right) + l^*(N \backslash (\lambda M)) &= l^*\left(\lambda\left((\lambda^{-1}N) \cap M\right)\right) + l^*\left(\lambda\left((\lambda^{-1}N) \backslash M\right)\right) \\ &= |\lambda|\, l^*\left((\lambda^{-1}N) \cap M\right) + |\lambda|\, l^*\left((\lambda^{-1}N) \backslash M\right) \\ &= |\lambda|\, l^*\left(\lambda^{-1}N\right) = l^*(N) \end{aligned}$$

so $\lambda M \in \mathcal{L}(\mathbb{R})$.

**Step 3: Conclusion:**

$l$ inherit the properties of $l^*$ from Proposition III.3 on $\mathcal{L}(\mathbb{R})$ since $l = l^*_{|\mathcal{L}(\mathbb{R})}$.

**Theorem III.9:** *Translation invariant and scaling characterization of $l$*

Let $\mu$ be a measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$, with $\mu([0,1]) < +\infty$,

1. If $\mu$ is translation invariant, then $\mu = \mu([0,1]) \cdot l$
2. If $\forall M \in \mathcal{L}(\mathbb{R}), \forall \lambda \in \mathbb{R}, \mu(\lambda M) = |\lambda|\, \mu(M)$ then $\mu([0,1]) \cdot l$.

Fixing $\mu([0,1])$ implies that $\mu = l$ in the above statements. Thus, $l$ is the only translation invariant measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ such that $\mu([0,1]) = 1$.

**Proof:**

1. Assume by contradiction that $\exists x \in E | \mu(\{x\}) > 0$ then $\mu = \mu(\{x\}) \cdot \#$ and $\mu([0,1]) = +\infty$. So $\forall x \in E, \mu(\{x\}) = 0$.

   Let $\frac{p}{q} \in \mathbb{Q}$,

   $$\mu([0,1]) = \sum_{n=0}^{q-1} \mu\left(\left[\frac{n}{q}, \frac{n+1}{q}\right]\right) = q\mu\left(\left[0, \frac{1}{q}\right]\right)$$

   and similarly

   $$\mu\left(\left[0, \frac{p}{q}\right]\right) = p\mu\left(\left[0, \frac{1}{q}\right]\right) = \frac{p}{q}\mu([0,1])$$

   Let $x \in \mathbb{R}$ and $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ a non-increasing sequence such converging to $x$,

   $$\mu([0,x]) = \mu\left(\bigcap_{n\in\mathbb{N}}\left[0, \frac{p_n}{q_n}\right]\right) = \lim_{n\to\infty} \mu\left(\left[0, \frac{p_n}{q_n}\right]\right) = \lim_{n\to\infty} \frac{p_n}{q_n}\mu([0,1]) = x\mu([0,1])$$

   if $\mu([0,1]) = 0$ then $\mu = 0$ and the result is satisfied, otherwise $\mu([0,1])^{-1}\mu$ extends the length of intervals to by Theorem III.7, $\mu([0,1])^{-1}\mu = l$.

2. From

   $$\begin{aligned}\mu(\{0\}) + 2\mu(\,]0,1]) &= \mu(\{0\}) + \mu(\,]0,2]) = \mu([0,2]) = 2\mu([0,1]) \\ &= 2\left(\mu(\{0\}) + \mu(\,]0,1])\right)\end{aligned}$$

   it follows that $\mu(\{0\}) = 0$.

   Assume by contradiction that $\exists a \in \mathbb{R}^* | \mu(\{a\}) > 0$, then $\forall x \in \mathbb{R}$,

   $$\mu(\{x\}) = |x|\,\mu(\{1\}) = \left|\frac{x}{a}\right|\mu(\{a\})$$

   so $\mu([0,1]) = +\infty$. Thus, $\forall x \in \mathbb{R}, \mu(\{x\}) = 0$.

   Let $x \in \mathbb{R}_+$,

   $$\mu([0,x]) = x\mu([0,1] = \mu([-x,0])$$

   so

   $$\begin{aligned}&\text{if } 0 \leqslant a \leqslant b, \text{ then } \mu([a,b]) = \mu([0,b]) - \mu([0,a]) = (b-a)\mu([0,1]) \\ &\text{if } 0 \leqslant a \leqslant b, \text{ then } \mu([a,b]) = \mu([a,0]) + \mu([0,b]) = (-a+b)\mu([0,1]) \\ &\text{if } 0 \leqslant a \leqslant b, \text{ then } \mu([a,b]) = \mu([a,0]) - \mu([b,0]) = (-a-(-b))\mu([0,1])\end{aligned}$$

   The conclusion is then the same than for the translation invariant case.

# IV Measurable functions

Let $(E, \mathcal{A})$ and $(F, \beta)$ be two measurable sets.

**Definition IV.1:** *Measurable function*

A function $f : (E, \mathcal{A}) \to (F, \beta)$ is called measurable is $\forall M \in \beta, f^{-1}(M) \in \mathcal{A}$

This is analogous to continuous functions as $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $\forall \mathcal{O} \subseteq \mathbb{R}$ open, $f^{-1}(\mathcal{O})$ is open.

Let $M \in \mathcal{P}(E)$, $\mathbb{1}_M : (E, \mathcal{A}) \to (\{0,1\}, \mathcal{P}(\{0,1\}))$ is measurable (as a function) if and only if $M \in \mathcal{A}$ ($M$ is measurable as a set). This follows from $\mathbb{1}_M^{-1}(\{1\}) = M$ and $\mathbb{1}_M^{-1}(\{0\}) = M^c$.

Next we check that the measurability of a function is unchanged if one restrict the codomain of the function to its range.

**Proposition IV.2**

$f : (E, \mathcal{A}) \to (F, \beta)$ is measurable if and only if $f : (E, \mathcal{A}) \to \left(\text{ran}(f), \beta_{\text{ran}(f)}\right)$

**Proof:**

Recalling that

$$\beta_{\text{ran}(f)} = \{M \cap \text{ran}(f), M \in \beta\}$$

the result follows from the fact that $\forall M \in \beta$,

$$f^{-1}(M) = f^{-1}(M \cap \text{ran}(f))$$

We only need to check measurability on the generators of a $\sigma$-algebra:

**Proposition IV.3**

Let $\mathcal{M} \subseteq F$, $f : (E, \mathcal{A}) \to (F, \sigma(\mathcal{M}))$ is measurable if and only if $\forall M \in \mathcal{M}, f^{-1}(M) \in \mathcal{A}$

**Proof:**

Assume that $\forall M \in \mathcal{M}, f^{-1}(M) \in \mathcal{A}$, we define

$$\mathcal{A}_f := \left\{M \in \sigma(\mathcal{M}) | f^{-1}(M)\right\}$$

By construction, $\mathcal{M} \subseteq \mathcal{A}_f$. Then we proove that $\mathcal{A}_f$ is a $\sigma$-algebra:

- $f^{-1}(F) = E \in \mathcal{A}$ so $E \in \mathcal{A}_f$
- Let $M \in \mathcal{A}_f$, $f^{-1}(M^c) = f^{-1}(M)^c \in \mathcal{A}$ so $M^c \in \mathcal{A}_f$
- Let $(M_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}_f$, then

$$f^{-1}\left(\bigcup_{n \in \mathbb{N}} M_n\right) = \bigcup_{n \in \mathbb{N}} f^{-1}(M_n) \in \mathcal{A}$$

sp

$$\bigcup_{n\in\mathbb{N}} M_n \in \mathcal{A}_f$$

This implies that $\sigma(M) \subseteq \mathcal{A}_f$ hence $\sigma(M) = \mathcal{A}_f$ meaning that $f$ is measurable. The contraposition simply follows from the fact that $\mathcal{M} \subseteq \sigma(\mathcal{M})$.

As a corollary, all continuous functions are measurable with respect to the Borel $\sigma$-algebras. Indeed a Borel $\sigma$-algebra if generated by open sets and the pre-imagine of an open set by a continuous function is an open set.

**Proposition IV.4**

1. Let $f : (E, \mathcal{A}) \to (F, \beta)$ and $g : (F, \beta) \to (G, \Gamma)$ be two measurable functions, then $g \circ f : (E, \mathcal{A}) \to (G, \Gamma)$ is measurable.
2. Let $f_n : (E, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a sequence of measurable functions, then

$$\sup_{n\in\mathbb{N}} f_n,\ \inf_{n\in\mathbb{N}} f_n,\ \limsup_{n\to\infty} f_n,\ \liminf_{n\to\infty} f_n$$

are measurable.

**Proof:**

1. Let $M \in \Gamma$ then $(g \circ f)^{-1}(M) = f^{-1}(\underbrace{g^{-1}(M)}_{\in\beta}) \in \mathcal{A}$.
2. Using Proposition IV.3 and

$$\mathcal{B}(\mathbb{R}) = \sigma(\{]a, +\infty[, a \in \mathbb{R}\})$$

we only need to prove that $\forall a \in \mathbb{R}$,

$$\begin{aligned}\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}(\ ]a, +\infty[\ ) = \left\{x \in E \mid \sup_{n\in\mathbb{N}} f_n(x) > a\right\} &= \{x \in E \mid \exists n \in \mathbb{N} \mid f_n(x) > a\} \\ = \bigcup_{n\in\mathbb{N}} \{x \in E \mid f_n(x) > a\} &= \bigcup_{n\in\mathbb{N}} \underbrace{f_n^{-1}(\ ]a, +\infty[\ )}_{\in\mathcal{A}} \in \mathcal{A}\end{aligned}$$

The same holds for the infimum using

$$\mathcal{B}(\mathbb{R}) = \sigma(\{] - \infty, a[, a \in \mathbb{R}\})$$

Then we apply this to

$$\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} \sup_{k\geqslant n} f_k, \quad \liminf_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} f_k$$

**Theorem IV.5:** *Limit of measurable functions*

Let $(X, d)$ be a metric space and $f_n : (E, \mathcal{A}) \to (X, \mathcal{B}(X))$ a sequence of measurable functions. If $f_n \to f$ pointwise, then $f$ is measurable.

**Proof:**

Let $\mathcal{O}$ be an open set of $X$. We introduce

$$g : \begin{array}{ccc} X & \to & \mathbb{R} \\ x & \to & d(x, \mathcal{O}^c) := \inf_{y \in \mathcal{O}^c} d(x, y) \end{array}$$

Denote $\mathcal{O}_n := g^{-1}\left( \left]\frac{1}{n}, \infty\right[ \right)$,

**Step 1:** $\mathcal{O} = \bigcup_{n \in \mathbb{N}} \mathcal{O}_n$

$\boxed{\subseteq}$ Let $x \in \mathcal{O}$. $\mathcal{O}$ is open so $\exists n \in \mathbb{N}^*$ such that

$$B\left(x, \frac{1}{n}\right) \subseteq \mathcal{O}$$

Thus $\forall y \in \mathcal{O}^c, d(x, y) \geqslant \frac{1}{n}$ and $g(x) \geqslant \frac{1}{n} > \frac{1}{n+1}$ meaning that $x \in \mathcal{O}_{n+1}$.

$\boxed{\supseteq}$ $x \in \mathcal{O}^c \implies g(x) = 0$ so

$$x \in \bigcup_{n \in \mathbb{N}^*} \mathcal{O}_n \implies \exists n \in \mathbb{N}^* \mid g(x) > \frac{1}{n} \implies x \in \mathcal{O}$$

**Step 2:** $g$ **is continuous**

$\forall z \in X,\ d(x, z) \leqslant d(x, y) + d(y, z)$ hence taking the infimum over $z \in \mathcal{O}^c$ we obtain

$$g(x) \leqslant d(x, y) + g(y)$$

thus

$$g(x) - g(y) \leqslant d(x, y)$$

Exchanging the rôles of $x$ and $y$ we get

$$g(y) - g(x) \leqslant d(y, x) = d(x, y)$$

so

$$|g(x) - g(y)| \leqslant d(x, y)$$

**Step 3:** $M \in \mathcal{A}$

From the previous steps we know that $\mathcal{O}_n$ is open and that

$$f^{-1}(\mathcal{O}) = \bigcup_{n \in \mathbb{N}} f^{-1}(\mathcal{O}_n)$$

so

$$x \in f^{-1}(\mathcal{O}) \iff \exists n \in \mathbb{N}^* \mid \lim_{k\to\infty} f_k(x) = f(x) \in \mathcal{O}_n$$
$$\iff \exists n \in \mathbb{N}^*,\ \exists m \in \mathbb{N} \mid \forall k \geqslant m,\ f_k(x) \in \mathcal{O}_n$$

therefore

$$f^{-1}(\mathcal{O}) = \bigcup_{n\in\mathbb{N}^*} \bigcup_{m\in\mathbb{N}} \bigcap_{k\geqslant m} \underbrace{f_k^{-1}(\mathcal{O}_n)}_{\in\mathcal{A} \text{ since } \mathcal{O}_n \text{ open}} \in \mathcal{A}$$

**Definition IV.6:** *simple functions*

A simple function is a function of the form

$$s \coloneqq \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k}$$

with $n \in \mathbb{N}$, $(\lambda_k)_{[\![0,n]\!]} \subseteq \mathbb{R}^*$ and $(M_k)_{k\in[\![0,n]\!]} \subseteq \mathcal{A}$ disjoints.

**Proposition IV.7**

Simple functions are measurable and closed under sum and product.

**Proof:**

Let

$$s \coloneqq \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k}$$

be a simple function on $(E, \mathcal{A})$. $s$ is valued in $\{0\} \cup \{\lambda_k, [\![0,n]\!]\}$ and

$$s^{-1}(\{0\}) = \left(\bigcup_{k=0}^{n} M_i\right)^c \in \mathcal{A}$$
$$\forall\, [\![0,n]\!],\ s^{-1}(\{\lambda_k\}) = M_k \in \mathcal{A}$$

so $s$ is measurable. Let

$$h \coloneqq \sum_{q=0}^{m} \mu_q \mathbb{1}_{N_q}$$

be another simple function on $(E, \mathcal{A})$ and $c \in \mathbb{R}_+$. Then

$$sh = \sum_{k=0}^{n}\sum_{q=0}^{m} \lambda_k\mu_q \mathbb{1}_{M_k}\mathbb{1}_{N_q} = \sum_{k=0}^{n}\sum_{q=0}^{m} \lambda_k\mu_q \mathbb{1}_{M_k\cap N_q}$$
$$s + h = \sum_{k=0}^{n}\sum_{q=0}^{m} (\lambda_k + \mu_q)\mathbb{1}_{M_k\cap N_q} + \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k\setminus \bigcup_{q=0}^{m} N_q} + \sum_{q=0}^{m} \mu_q \mathbb{1}_{N_q\setminus \bigcup_{k=0}^{n} M_k} \tag{IV.1}$$

Simple functions are also closed under multiplication by a non-negative constant since a non-negative constant is a simple function.

**Theorem IV.8**

Let $f : (E, \mathcal{A}) \to \overline{\mathbb{R}}_+$ measurable. Then $f$ is the pointwise increasing limit of non negative simple functions. If $f$ real valued, then it is the pointwise limit of simple functions.

**Proof:**

First, we assume that $f \geqslant 0$

**Step 1: Constructing a sequence**

Let $n \in \mathbb{N}^*$, we decompose

$$\overline{R}_+ = \left( \bigcup_{k=1}^{n2^n} \left[ \frac{k-1}{2^n}, \frac{k}{2^n} \right] \right) \cup [n, +\infty]$$

Let

$$A_{n,k} \coloneqq f^{-1}\left( \left[ \frac{k-1}{2^n}, \frac{k}{2^n} \right] \right) \qquad B_n \coloneqq f^{-1}([n, +\infty])$$

so that

$$E = \left( \bigcup_{k=1}^{n2^n} A_{n,k} \right) \cup B_n$$

we introduce

$$f_n \coloneqq \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbb{1}_{A_{n,k}} + n\mathbb{1}_{B_n}$$

**Step 2: $(f_n)_{n \in \mathbb{N}}$ is increasing**

- If $x \in A_{n,k}$. We start by remarking that

$$\left[ \frac{k-1}{2^n}, \frac{k}{2^n} \right[ = \left[ \frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}} \right[ \sqcup \left[ \frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}} \right[$$

so $A_{n,k} = A_{n+1,2k-1} \cup A_{n+1,2k}$ and either

$$f_{n+1}(x) = \frac{2k-2}{2^{n+1}} = \frac{k-1}{2^n} = f_n(x)$$

or

$$f_{n+1}(x) = \frac{2k-1}{2^{n+1}} > \frac{k-1}{2^n} = f_n(x)$$

- If $x \in B_n$ then either $x \in B_{n+1} \subseteq B_n$ and

$$f_{n+1}(x) = n + 1 > n = f_n(x)$$

or

$$f(x) \in [n, n+1[= \bigcup_{k=n2^{n+1}+1}^{(n+1)2^{n+1}} \left[\frac{k-1}{2^{n+1}}, \frac{k}{2^{n+1}}\right[$$

so $\exists k \in [\![n2^{n+1} + 1, (n+1)2^{n+1}]\!]$ such that $x \in A_{n+1,k}$ and

$$f_{n+1}(x) = \frac{k-1}{2^{2^{n+1}}} \geqslant n = f_n(x)$$

### Step 3: $f_n \to f$ pointwise

By construction $f_n \leqslant f$. Let $x \in E$.

- If $f(x) = +\infty$, then $\forall n \in \mathbb{N}^*$,

$$f_n(x) = n \underset{n\to\infty}{\longrightarrow} +\infty = f(x)$$

- Otherwise $\forall n > f(x)$, $\exists k \in [\![1, n2^n]\!] \mid f(x) \in \left[\!\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]\!\right]$ so $f_n(x) = \frac{k-1}{2^n}$ and

$$|f(x) - f_n(x)| \leqslant \frac{1}{2^n} \underset{n\to\infty}{\longrightarrow} 0$$

### Step 4: If $f$ real valued

Let $f_+ \coloneqq \max(f, 0) \geqslant 0$ and $f_- = -\min(f, 0) \geqslant 0$ so that $f = f_+ - f_-$. From the previous steps, there exists sequences of simple functions $(f_{n,+})_{n\in\mathbb{N}}, (f_{n,-})_{n\in\mathbb{N}}$ such that

$$f_{n,+} \underset{n\to\infty}{\longrightarrow} f_+ \quad f_{n,-} \underset{n\to\infty}{\longrightarrow} f_-$$

so

$$f_{n,+} - f_{n,-} \underset{n\to\infty}{\longrightarrow} f$$

and $f_{n,+} - f_{n,-}$ is simple due to Proposition IV.7.

**Proposition IV.9**

Let $f, g : (E, \mathcal{A}) \to \mathbb{R}$ measurable and $\lambda \in \mathbb{R}$ then $f + g,\ fg,\ \lambda f$ are measurable.

**Proof:**

From Theorem IV.8, there exists sequences of simple functions $(f_n)_{n\in\mathbb{N}},\ (g_n)_{n\in\mathbb{N}}$ such that

$$f_n \underset{n\to\infty}{\to} f \quad g_n \underset{n\to\infty}{\to} g$$

and thus

$$\begin{aligned} f_n + g_n &\underset{n\to\infty}{\to} f + g \\ f_n g_n &\underset{n\to\infty}{\to} fg \\ \lambda f_n &\underset{n\to\infty}{\to} \lambda f \end{aligned}$$

while $f_n + g_n,\ f_n g_n,\ \lambda f_n$ are measurable due to Proposition IV.7. We conclude with Theorem IV.5.

# V Lebesgue's integral

Let $(E, \mathcal{A}, \mu)$ be a measured space.

**Definition V.1:** *Lebesgue integral*

- Given $M \in \mathcal{A}$, we define

$$\int_E \mathbb{1}_M d\mu := \mu(M)$$

- Given a non-negative simple function

$$s := \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k}$$

we define

$$\int_E s d\mu := \sum_{k=0}^{n} \lambda_k \int_E \mathbb{1}_{M_k} d\mu$$

- Given $f : (E, \mathcal{A}, \mu) \to \overline{\mathbb{R}}_+$ a non-negative measurable function we define

$$\int_E f d\mu := \sup_{\substack{s \text{ simple} \\ 0 \leqslant s \leqslant f}} \int_E s d\mu$$

- We say that a measurable function $f : (E, \mathcal{A}, \mu) \to \overline{\mathbb{R}}$ is integrable if

$$\int_E |f| < +\infty$$

and in this case define

$$\int_E f d\mu := \int_E f_+ d\mu - \int_E f_- d\mu$$

- Given $M \in \mathcal{A}$, if $f \mathbb{1}_M$ is non-negative or integrable, we define

$$\int_M f d\mu := \int_E f \mathbb{1}_M d\mu$$

A few comments:

- Integral have to computed with the convention that $0 \cdot \infty = 0$

- The integral for a simple function is

$$\int_E s d\mu = \sum_{k=0}^{n} \lambda_k \mu(M_k)$$

- Taking $\mu = l$ leads to connections with the Riemann integral.
- Other notation:

$$\int_E f d\mu =: \int_E f(x) d\mu(x)$$

- We can extend the definition of the integral to non-integrable measurable functions $f : E \to \overline{\mathbb{R}}$ such that

$$\int_E f_+ d\mu = +\infty \quad \text{xor} \quad \int_E f_- d\mu = +\infty$$

  through

$$\int_E f d\mu := \int_E f_+ d\mu - \int_E f_- d\mu$$

**Proof:**

We need to check that the definition of the Lebesgue integral of a simple function is independent of the choice of measurable sets. Assume that

$$s = \sum_{k=1}^{n} \lambda_k \mathbb{1}_{M_k} = \sum_{q=1}^{m} \mu_q \mathbb{1}_{N_q}$$

Denote

$$M_0 := E \backslash \bigcup_{k=1}^{n} M_k, \quad N_0 := E \backslash \bigcup_{q=1}^{m} N_q, \quad \lambda_0 := \mu_0 := 0$$

we still have that

$$s = \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k} = \sum_{q=0}^{m} \mu_q \mathbb{1}_{N_q}$$

and then $\forall k \in [\![0, n]\!], \forall q \in [\![0, m]\!], \ M_k \cap N_q \neq \varnothing \implies \lambda_k = \mu_q$ so

$$\begin{aligned}\sum_{k=0}^{n} \lambda_k \mu(M_k) &= \sum_{k=0}^{n} \lambda_k \mu\left(M_k \cap \bigcup_{q=0}^{m} N_q\right) = \sum_{k=0}^{n}\sum_{q=0}^{m} \lambda_k \mu(M_k \cap N_q) = \sum_{k=0}^{n}\sum_{q=0}^{m} \mu_q \mu(M_k \cap N_q) \\ &= \sum_{q=0}^{m} \mu_q \mu\left(N_q \cap \bigcup_{k=0}^{n} M_k\right) = \sum_{q=0}^{m} \mu_q \mu(N_q)\end{aligned}$$

hence

$$\sum_{k=1}^{n} \lambda_k \mu(M_k) = \sum_{q=1}^{m} \mu_q \mu(N_q)$$

then

**Example V.2:** *The Dirac mass*

If $a \in E$ and $\mu \coloneqq \delta_a$ then

$$\int_E f d\delta_a = f(a)$$

- Indicator functions: let $M \in \mathcal{A}$, then

$$\int_E \mathbb{1}_M d\delta_a = \delta_a(M) = \mathbb{1}_M(a)$$

- Simple functions: let

$$s \coloneqq \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k}$$

be a non-negative simple function. Then

$$\int_E s d\delta_a = \sum_{k=0}^{n} \lambda_k \delta_a(M_k) = \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k}(a) = s(a)$$

- Non negative measurable functions: let $f : E \to \overline{R}_+$ measurable, then

$$\int_E f d\delta_a = \sup_{\substack{s \text{ simple} \\ 0 \leqslant s \leqslant f}} s(a) \leqslant f(a)$$

But then we have equality in the above by choosing $s \coloneqq f(a)\mathbb{1}_a$.

- Real valued measurable functions: let $f : E \to \overline{R}_+$ measurable, since

$$\int_E |f| \, d\delta_a = |f(a)|$$

we see that $f$ is integrable if and only if $|f(a)| < +\infty$ and in this case

$$\int_E f d\delta_a = \int_E f_+ d\delta_a - \int_E f_- d\delta_a = f_+(a) - f_-(a) = f(a)$$

**Proposition V.3**

Let $f, g : (E, \mathcal{A}, \mu) \to \overline{\mathbb{R}}$ measurable, the following hold as soon as the integrals are defined:

1. Markov's inequality: $\forall a > 0$,

$$\mu\left(f^{-1}([a, +\infty])\right) \leqslant \frac{1}{a} \int_E |f| \, d\mu$$

2. If $\mu(E) = 0$ or $f = 0$, then

$$\int_E f d\mu = 0$$

3. The integral is non-decreasing: let $M, N \in \mathcal{A}$,

$$\bullet\ f \leqslant g \implies \int_E f d\mu \leqslant \int_E g d\mu$$

$$\bullet\ M \subseteq N, f \geqslant 0 \implies \int_M f d\mu \leqslant \int_N f d\mu$$

**Proof:**

1. $|f|$ is measurable as composition of $f$ and $|\cdot|$ and $a\mathbb{1}_{f^{-1}([a,+\infty])}$ is a non-negative simple function such that

$$a\mathbb{1}_{f^{-1}([a,+\infty])} \leqslant |f|$$

therefore

$$a\mu(f^{-1}([a, +\infty])) \leqslant \int_E |f| \, d\mu$$

2. If $\mu(E) = 0$, then $\forall M \in \mathcal{A}, \mu(M) = 0$ so

$$\int_E \mathbb{1}_M d\mu = \mu(M) = 0$$

thus the same follows for simple functions and $f$, $f$ being integrable as

$$\int_E |f| \, d\mu = 0$$

Next, we remark that the 0 function can be written as $0 = \mathbb{1}_\varnothing$ so

$$\int_E 0 d\mu = \int_E \mathbb{1}_\varnothing d\mu = \mu(\varnothing) = 0$$

3. Assume that $0 \leqslant f \leqslant g$ and let $s$ be a non-negative simple function. Since $s \leqslant f \implies s \leqslant g$,

$$\{s \text{ simple } | 0 \leqslant s \leqslant f\} \subseteq \{s \text{ simple } | 0 \leqslant s \leqslant g\}$$

so

$$\int_E f d\mu \leqslant \int_E g d\mu$$

Now, assume that $f, g : E \to \overline{\mathbb{R}}$ are integrable such that $f \leqslant g$, then $f_+ \leqslant g_+$ and $f_- \geqslant g_-$ so

$$\int_E d\mu = \int_E f_+ d\mu - \int_E f_- d\mu \leqslant \int_E g_+ d\mu - \int_E g_- d\mu = \int_E g d\mu$$

As a consequence if $M \subseteq N$ and $f \geqslant 0$, then $f\mathbb{1}_M \leqslant f\mathbb{1}_N$ and

$$\int_M f d\mu = \int_E f\mathbb{1}_M d\mu \leqslant \int_E f\mathbb{1}_N d\mu = \int_N f d\mu$$

**Theorem V.4:** *Monotone convergence*

Let $f_n : (E, \mathcal{A}, \mu) \to \overline{\mathbb{R}}_+$ be a non decreasing sequence of measurable functions, then

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E \lim_{n\to\infty} f_n d\mu$$

**Proof:**

**Step 1: A first lemma**

Let $N \in \mathcal{A}$, we remark that

$$\mu_N : \begin{array}{ccc} \mathcal{A} & \to & \mathbb{R} \\ M & \mapsto & \mu(N \cap M) \end{array}$$

is a measure as

- $\mu_N(\varnothing) = \mu(\varnothing) = 0$
- If $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ disjoints, then $(N \cap M_n)_{n\in\mathbb{N}}$ are also disjoints so

$$\begin{aligned}\mu_N\left(\bigcup_{n\in\mathbb{N}} M_n\right) &= \mu\left(N \cap \bigcup_{n\in\mathbb{N}} M_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}}(N \cap M_n)\right) = \sum_{n\in\mathbb{N}} \mu(N \cap M_n) \\ &= \sum_{n\in\mathbb{N}} \mu_N(M_n)\end{aligned}$$

Let

$$s := \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k}$$

be a simple function, then

$$\mu_s : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathbb{R} \\ M & \longmapsto & \int\limits_M s d\mu \end{array}$$

is a also a measure, as sum of measures, since

$$\int\limits_M s d\mu = \int\limits_E s \mathbb{1}_M d\mu = \int\limits_E \left( \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M_k} \mathbb{1}_M \right) d\mu = \int\limits_E \left( \sum_{k=0}^{n} \lambda_k \mathbb{1}_{M \cap M_k} \right) d\mu = \sum_{k=0}^{n} \lambda_k \mu(M \cap M_k)$$
$$= \sum_{k=0}^{n} \lambda_k \mu_M(M_k)$$

### Step 2: A second lemma

Let $t \in \mathbb{R}$, then with the notations from the previous lemma,

$$ts = \sum_{k=0}^{n} t \lambda_k \mathbb{1}_{M_k}$$

is a simple function so

$$\int\limits_E ts d\mu = \sum_{k=0}^{n} t \lambda_k \mu(M_k) = t \sum_{k=0}^{n} \lambda_k \mu(M_k) = t \int\limits_E s d\mu$$

### Step 3: Proof

let $f$ be the pointwise limit of $(f_n)_{n \in \mathbb{N}}$.

$\boxed{\leqslant}$ As $\forall n \in \mathbb{N},\ f_n \leqslant f$, by monotonicity of the integral,

$$\int\limits_E f_n d\mu \leqslant \int\limits_E f d\mu$$

Taking the limit $n \to \infty$, we obtain

$$\lim_{n \to \infty} \int\limits_E f_n d\mu \leqslant \int\limits_E f d\mu$$

$\boxed{\geqslant}$ Let $s$ be a simple function such that $0 \leqslant s \leqslant f$. Our goal is to prove that

$$\int\limits_E s d\mu \leqslant \lim_{n \to \infty} \int\limits_E f_n d\mu$$

so that we obtain the result by taking the supremum over $s$.

Let $t \in [0,1[$ and denote $\forall n \in \mathbb{N}$,

$$M_{n,t} \coloneqq \{x \in E | ts(x) \leqslant f_n(x)\} \in \mathcal{A}$$

which is a non-decreasing sequence in $n$. Let $x \in E$.

- If $f(x) = 0$, then $\forall n \in \mathbb{N},\ \forall t \in [0,1[,\ x \in M_{n,t}$
- Otherwise, $ts(x) \leqslant tf(x) < f(x)$ and up to a rank in $n$, $ts(x) \leqslant f_n(x)$

Thus

$$\bigcup_{n \in \mathbb{N}} M_{n,t} = E$$

and

$$\int\limits_{M_{n,t}} s d\mu = \mu_s(M_{n,t}) \underset{n\to\infty}{\longrightarrow} \mu_s\left(\bigcup_{n\in\mathbb{N}} M_{n,t}\right) = \mu_s(E) = \int\limits_E s d\mu$$

By definition of $M_{n,t}$, $\forall n \in \mathbb{N}$,

$$t \int\limits_{M_{n,t}} s d\mu = \int\limits_{M_{n,t}} st d\mu \leqslant \int\limits_{M_{n,t}} f_n d\mu \leqslant \int\limits_E f_n d\mu$$

Taking the limit $n \to \infty$, we get

$$t \int\limits_E s d\mu \leqslant \lim_{n\to\infty} \int\limits_E f_n d\mu \tag{V.1}$$

and we conclude by taking the limit $t \to 1$.

**Proposition V.5**

Let $f, g : (E, \mathcal{A}, \mu) \to \overline{\mathbb{R}}$ be measurable functions, then, as long as the integrals make sense, we have the followings properties:

1. Linearity: $\forall a, b \in \mathbb{R}$

$$\int\limits_E (af + bg) d\mu = a \int\limits_E f d\mu + b \int\limits_E g d\mu$$

2. Chasles relation: $\forall M, N \in \mathcal{A}$ disjoints,

$$\int\limits_{M \cup N} f d\mu = \int\limits_M f d\mu + \int\limits_N f d\mu$$

A useful consequence is that if $f = g$ $\mu$-a.e., meaning that $\exists N \in \mathcal{A}$ such that $\mu(N) = 0$ and $f_{|N^c} = g_{|N^c}$, we have that

$$\int_E f d\mu = \int_{N^c} f d\mu + \underbrace{\int_N f d\mu}_{=0} = \int_{N^c} g d\mu + \underbrace{\int_N g d\mu}_{=0} = \int_E g d\mu$$

The implication of this is that all the properties of the Lebesgue integral are true if the functions are only defined $\mu$-a.e. For example, if $f \leqslant g$ $\mu$-a.e., then

$$\int_E f d\mu \leqslant \int_E g d\mu$$

**Proof:**

1. Multiplication by a scalar property: let $\lambda \in \mathbb{R}$.

    - If $\lambda > 0$, then for $f \geqslant 0$, using (V.1) with the change of variable $\tilde{s} := \frac{s}{\lambda}$,

    $$\int_E \lambda f d\mu = \sup_{\substack{s \text{ simple} \\ 0 \leqslant s \leqslant \lambda f}} \int s d\mu = \sup_{\substack{\tilde{s} \text{ simple} \\ 0 \leqslant \tilde{s} \leqslant f}} \int \lambda \tilde{s} d\mu = \lambda \sup_{\substack{\tilde{s} \text{ simple} \\ 0 \leqslant \tilde{s} \leqslant f}} \int \tilde{s} d\mu = \lambda \int_E f d\mu$$

    For $f$ real valued, as $(\lambda f)_+ = \lambda f_+$, $(\lambda f)_- = \lambda f_-$,

    $$\int_E \lambda f d\mu = \int_E \lambda f_+ d\mu - \int_E \lambda f_- d\mu = \lambda \int_E f_+ d\mu - \lambda \int_E f_- d\mu = \lambda \int_E f d\mu$$

    - If $\lambda = 0$, then

    $$\int_E \lambda f d\mu = 0 = \lambda \int_E f d\mu$$

    - If $\lambda < 0$, for $f \geqslant 0$, $(\lambda f)_+ = 0$, $(\lambda f)_- = -\lambda f$ so

    $$\int_E \lambda f d\mu = \int_E (\lambda f)_+ d\mu - \int_E (\lambda f)_- d\mu = -\int_E \underbrace{(-\lambda)}_{\geqslant 0} f d\mu = -(-\lambda) \int_E f d\mu = \lambda \int_E f d\mu$$

    For $f$ real valued, $(\lambda f)_+ = -\lambda f_-$, $(\lambda f)_- = -\lambda f_+$ so

    $$\int_E \lambda f d\mu = \int_E (-\lambda) f_- d\mu - \int_E (-\lambda) f_+ d\mu = -\lambda \int_E f_- d\mu + \lambda \int_E f_+ d\mu = \lambda \int_E f d\mu$$

    Sum property:

    - For simple functions, starting with the notation from (IV.1), by $\sigma$-additivity,

    $$\int_E (s+h) d\mu = \sum_{k=0}^{n} \sum_{q=0}^{m} (\lambda_k + \mu_q) \mu\left(M_k \cap N_q\right) + \sum_{k=0}^{n} \lambda_k \mu\left(M_k \setminus \bigcup_{q=0}^{m} N_q\right)$$

$$
\begin{aligned}
&+\sum_{q=0}^{m}\mu_q\mu\left(N_q\setminus\bigcup_{k=0}^{n}M_k\right)\\
&=\sum_{k=0}^{n}\lambda_k\mu\left(M_k\cap\bigcup_{q=0}^{m}N_q\right)+\sum_{k=0}^{n}\lambda_k\mu\left(M_k\setminus\bigcup_{q=0}^{m}N_q\right)\\
&+\sum_{q=0}^{m}\mu_q\mu\left(N_q\cap\bigcup_{k=0}^{n}M_k\right)+\sum_{q=0}^{m}\mu_q\mu\left(N_q\setminus\bigcup_{k=0}^{n}M_k\right)\\
&=\sum_{k=0}^{n}\lambda_k\mu(M_k)+\sum_{q=0}^{m}\mu_q\mu(N_q)=\int_E s d\mu+\int_E h d\mu
\end{aligned}
$$

- For $f, g \geqslant 0$, there exists non negative, non decreasing sequences of simple functions $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ such that $f_n \underset{n\to\infty}{\to} f$, $g_n \underset{n\to\infty}{\to} g$. Then, by monotone convergence,

$$
\int_E (f+g)d\mu=\lim_{n\to\infty}\int_E (f_n+g_n)d\mu=\lim_{n\to\infty}\int_E f_n d\mu+\lim_{n\to\infty}\int_E g_n d\mu=\int_E f d\mu+\int_E g d\mu
$$

- For $f, g$ real valued,

$$
\begin{aligned}
&\begin{cases} f+g=(f+g)_+-(f+g)_- \\ f+g=f_+-f_-+g_+-g_- \end{cases}\\
&\implies (f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+\\
&\implies \int_E (f+g)_+ d\mu+\int_E f_- d\mu+\int_E g_- d\mu=\int_E (f+g)_- d\mu+\int_E f_+ d\mu+\int_E g_+ d\mu\\
&\implies \int_E (f+g)_+ d\mu-\int_E (f+g)_- d\mu=\int_E f_+ d\mu-\int_E f_- d\mu+\int_E g_+ d\mu-\int_E g_- d\mu\\
&\implies \int_E (f+g)d\mu=\int_E f d\mu+\int_E g d\mu
\end{aligned}
$$

2. As a consequence of linearity,

$$
\begin{aligned}
\int_{M\cup N} f d\mu &= \int_E f(\mathbb{1}_{M\cup N})d\mu=\int_E f(\mathbb{1}_M+\mathbb{1}_N)d\mu=\int_E f\mathbb{1}_M d\mu+\int_E f\mathbb{1}_N d\mu\\
&=\int_M f d\mu+\int_N f d\mu
\end{aligned}
$$

**Lemma V.6:** *Fatou*

Let $f_n : (E, \mathcal{A}, \mu) \to \overline{\mathbb{R}}_+$ be a measurable function, then

$$\int_E \liminf_{n\to\infty} f_n d\mu \leqslant \liminf_{n\to\infty} \int_E f_n d\mu$$

**Proof:**

$\forall n \in \mathbb{N}, \ \inf_{k\geqslant n} f_k \leqslant f_n$ so

$$\int_E \inf_{k\geqslant n} f_k d\mu \leqslant \int_E f_n d\mu$$

thus

$$\liminf_{n\to\infty} \int_E \inf_{k\geqslant n} f_k d\mu \leqslant \liminf_{n\to\infty} \int_E f_n d\mu$$

But, the left term is non-decreasing, so by monotone convergence

$$\liminf_{n\to\infty} \int_E \inf_{k\geqslant n} f_k d\mu = \lim_{n\to\infty} \int_E \inf_{k\geqslant n} f_k d\mu = \int_E \lim_{n\to\infty} \inf_{k\geqslant n} f_k d\mu = \int_E \liminf_{n\to\infty} f_n d\mu$$

which concludes.

**Theorem V.7:** *Dominated convergence*

Let $f_n : (E, \mathcal{A}, \mu) \to \overline{\mathbb{R}}$ be a sequence of measurable functions. If

- $f_n \underset{n\to\infty}{\overset{\mu\text{-a.e.}}{\longrightarrow}} f : E \to \overline{\mathbb{R}}$
- $\exists g : E \to \overline{\mathbb{R}}$ $\mu$-integrable such that $\forall n \in \mathbb{R}, \ |f_n| \leqslant g$ $\mu$-a.e.

then $f$ is integrable and

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$$

**Proof:**

Denote $\forall n \in \mathbb{N}, h_n \coloneqq 2g - |f - f_n|$, then since

$$|f - f_n| \leqslant |f| + |f_n| \leqslant 2g$$

so $h_n \geqslant 0$ $\mu$-a.e. With Fatou's lemma,

$$\int_E \liminf_{n\to\infty} h_n d\mu = 2\int_E g d\mu \leqslant \liminf_{n\to\infty} \int_E h_n d\mu = 2\int_E g d\mu + \liminf_{n\to\infty} \left( -\int_E |f - f_n| \, d\mu \right)$$
$$= 2\int_E g d\mu - \underbrace{\limsup_{n\to\infty} \int_E |f - f_n| \, d\mu}_{\leqslant 0}$$

so

$$0 \leqslant \lim_{n\to\infty} \int_E |f - f_n| \, d\mu \leqslant \limsup_{n\to\infty} \int_E |f - f_n| \, d\mu \leqslant 0$$

We conclude with the monotonicity of the integral:

$$\left| \int_E f_n d\mu - \int_E f d\mu \right| = \left| \int_E (f_n - f) d\mu \right| \leqslant \int_E |f_n - f| \, d\mu \underset{n\to\infty}{\longrightarrow} 0$$

# VI Applications and connections

**Proposition VI.1:** *Link with Riemann integral*

Let $f \in C^0([a,b],\mathbb{R})$, then

$$\int_a^b f(x)dx = \int_{[a,b]} f(x)dl(x)$$

Note that the left hand site of the equality is understood as the Riemann integral and the right and side as the Lebesgue integral with respect to the Lebesgue measure.

**Proof:**

First assume that $f \geqslant 0$ and for simplicity that $[a,b] = [0,1]$. For $n \in \mathbb{N}^*$ and $k \in [\![1, 2^n]\!]$, denote

$$I_{n,k} := \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right[$$

From Riemann integration theory, we know that

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{2^n}\sum_{k=1}^{2^n} \min_{I_{n,k}} f = \lim_{n\to\infty} \sum_{k=1}^{2^n} \left(\min_{I_{n,k}} f\right) l(I_{n,k}) = \lim_{n\to\infty} \int_{[0,1]} s_n dl$$

with

$$s_n := \sum_{k=1}^{2^n} \left(\min_{I_{n,k}} f\right) \mathbb{1}_{I_{n,k}}$$

Noticing that

$$I_{n,k} = I_{n+1,2k-1} \cup I_{n+1,2k}$$

we see that $(s_n)_{n\in\mathbb{N}^*}$ is non-decreasing. Let $x \in [0,1[$ and $\epsilon > 0$, then

$$\forall n \in \mathbb{N}^*,\ \exists k_n \in [\![1, 2^n]\!] \mid x \in I_{n,k_n}$$

As $f$ is continuous,

$$\exists \delta > 0 \mid \forall y \in [0,1[,\ |y-x| \leqslant \delta \implies |f(y)-f(x)| \leqslant \epsilon$$

So for $n\frac{1}{2^n} \leqslant \delta$ when $n \geqslant \ln_2\left(\frac{1}{\delta}\right)$ we have that

$$0 \leqslant f(x) - s_n(x) = \max_{y\in I_{n,k}}(f(x)-f(y)) \leqslant \epsilon$$

Hence the conclusion follows by monotone convergence.
For real valued functions, the result is generalized by splitting the integral into positive and negative parts and using the linearity of the Lebesgue and the Riemann integrals.

This result can be generalized for a more general class of functions. Indeed it is still true if $f : [a, b] \to \mathbb{R}$ if Riemann integrable and bounded. Then one can prove that

- $l\left(\{x \in [a, b] \mid f \text{ is not continuous at } x\}\right) = 0$
- $\exists\ \tilde{f} : [a, b] \to \mathbb{R}$ measurable such that $\tilde{f} = f$ $l$-a.e
- $\tilde{f}$ is Lebesgue integrable and

$$\int_a^b f(x)dx = \int_{[a,b]} \tilde{f}(x)dl(x)$$

Results also holds for generalized Riemann integrals.

The next results are applications of the dominated convergence theorem.

In the following we consider a function

$$f : \begin{array}{ccc} (X, d) \times (E, \mathcal{A}, \mu) & \to & \overline{\mathbb{R}} \\ (x, y) & \mapsto & f(x, y) \end{array}$$

where $(X, d)$ is a metric space and $(E, \mathcal{A}, \mu)$ is a measured space. Then denote

$$\mathcal{F} : \begin{array}{ccc} (X, d) & \to & \mathbb{R} \\ x & \mapsto & \int_E f(x, y)d\mu(y) \end{array}$$

$\mathcal{F}$ is called a parameter-dependent integral, here the parameter is the variable $x$.

We introduce a notation for the sections of $f$: $\forall x \in X$,

$$f(x, \bullet) : \begin{array}{ccc} (E, \mathcal{A}, \mu) & \to & \overline{\mathbb{R}} \\ y & \mapsto & f(x, y) \end{array}$$

and $\forall y \in E$,

$$f(\bullet, y) : \begin{array}{ccc} (X, d) & \to & \overline{\mathbb{R}} \\ (x, y) & \mapsto & f(x, y) \end{array}$$

so that $\mathcal{F}(x) = \int_E f(x, \bullet)d\mu$.

**Theorem VI.2:** *Continuity of parameter-dependent integrals*

If

- $f$ is continuous with respect to $x$ $\mu$-a.e. on $E$:

$$\text{for } \mu\text{-a.e. } y \in E,\ f(\bullet, y) \in C^0(X, \mathbb{R})$$

- $f$ is measurable with respect to $y$ on $X$:

$$\forall x \in X,\ f(x, \bullet) \text{ is measurable}$$

- $\exists\ g : E \mapsto \overline{\mathbb{R}}_+$ $\mu$-integrable such that

$$\forall x \in X,\ \text{for } \mu\text{-a.e. } y \in E,\ |f(x,y)| \leqslant g(y)$$

then $\mathcal{F} \in C^0(X, \mathbb{R})$.

**Proof:**

Let $x \in X$, $\mathcal{F}(x)$ is well defined and real valued as $f(x, \bullet)$ is $\mu$-integrable due to the last assumption.
Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x_n \underset{n\to\infty}{\to} x$. Denote

$$\forall y \in E,\ f_n(y) \coloneqq f(x_n, y)$$

then for $\mu$-a.e. $y \in E$, $f_n(y) \underset{n\to\infty}{\to} f(x,y)$ as $f$ is continuous with respect to $x$. Finally,

$$\forall n \in \mathbb{N},\ |f_n| \leqslant g$$

so by the dominated convergence theorem

$$\lim_{n\to\infty} \mathcal{F}(x_n) = \lim_{n\to\infty} \int_E f_n(y) d\mu(y) = \int_E f(x,y) d\mu(y) = \mathcal{F}(x)$$

**Example VI.3:** *Continuity of the Euler gamma function*

We define $\forall\ x > 0$,

$$\Gamma(x) \coloneqq \int_{\mathbb{R}_+} y^{x-1} e^{-y} dy$$

this is a well-defined and positive function known for being a continuous extension of the factorial as is satisfies

$$\forall x > 0,\ \Gamma(x+1) = x\Gamma(x)$$

hence $\forall n \in \mathbb{N}^*,\ \Gamma(n) = (n-1)!$. We can use the previous theorem the verify that this is indeed a continuous function.
We prove a that $\Gamma$ is continuous on every interval $[a,b] \subseteq \mathbb{R}_+^*$ so it is continuous on $\mathbb{R}_+^*$. In this case we have

$$f : \begin{array}{ccc} \mathbb{R}_+^* \times \mathbb{R}_+ & \to & \overline{\mathbb{R}} \\ (x,y) & \mapsto & y^{x-1}e^{-y} \end{array}$$

To apply to previous result we verify that

- $f$ is continuous with respect to $x$ on $\mathbb{R}_+^*$ so $f$ is continuous with respect to $x$ $l$-a.e. on $\mathbb{R}_+$ ($f$ have a discontinuity at $y = 0$ if and only if $x < 1$, but $l(\{0\}) = 0$).

- $f$ is measurable with respect to $y$ on $[a, b]$ as it is continuous.
- $\forall x \in [a, b],\ \forall y \in \mathbb{R}_+$,

$$|f(x, y)| = y^{x-1}e^{-y} \leqslant y^{a-1}\mathbb{1}_{y<1} + y^{b-1}e^{-y}\mathbb{1}_{y\geqslant 1} =: g(y)$$

  where $g$ is integrable on $\mathbb{R}_+$.

**Theorem VI.4:** *Differentiability of parameter-dependent integrals*

Assume that $X = ]a, b[$ and

- $f$ is measurable with respect to $y$ on $]a, b[$
- $\exists x_0 \in ]a, b[ \mid f(\bullet, y_0)$ is $\mu$-integrable
- $f$ is differentiable with respect to $x$ $\mu$-a.e. on $E$
- $\exists\ g : E \mapsto \overline{\mathbb{R}}_+$ $\mu$-integrable such that

$$\forall x \in ]a, b[,\ \text{for } \mu\text{-a.e. } y \in E,\ \left|\frac{\partial f}{\partial x}(x, y)\right| \leqslant g(y)$$

then $f$ is integrable with respect to $y$ on $]a, b[$, and $\mathcal{F}$ is differentiable with

$$\forall x \in ]a, b[\ , \mathcal{F}'(x) = \int_E \frac{\partial f}{\partial x}(x, y) d\mu(y)$$

**Proof:**

By the mean value theorem, $\forall x \in ]a, b[$, for $\mu$-a.e. $y \in E$,

$$|f(x, y) - f(x_0, y)| \leqslant |x - x_0|\, g(y) \leqslant (b - a)g(y)$$

thus by the triangular inequality,

$$|f(x, y)| \leqslant (b - a)g(y) + f(x_0, y) \tag{VI.1}$$

and as the right hand side is $\mu$-integrable in $y$, $f(x, \bullet)$ is $\mu$-integrable.
Let $x, \in ]a, b[$ and $(x_n)_{n\in\mathbb{N}} \subseteq ]a, b[ \setminus \{x\}$, such that $x_n \underset{n\to\infty}{\to} x$, then by linearity of the integral,

$$\frac{\mathcal{F}(x) - \mathcal{F}(x_n)}{x - x_n} = \int_E \underbrace{\frac{f(x, y) - f(x_n, y)}{x - x_n}}_{:= h_n(y)\ \text{(measurable)}} d\mu(y)$$

As $f$ is differentiable with respect to $x$, for $\mu$-a.e. $y \in E$,

$$h_n(y) \underset{n\to\infty}{\to} \frac{\partial f}{\partial x}(x, y)$$

Using again the mean value theorem, we get that $|h_n| \leqslant g$ $\mu$-a.e., so by the dominated convergence theorem, $\forall x \in ]a, b[$,

$$\mathcal{F}'(x) = \lim_{n\to\infty} \frac{\mathcal{F}(x) - \mathcal{F}(x_n)}{x - x_n} = \int_E \lim_{n\to\infty} \frac{f(x,y) - f(x_n,y)}{x - x_n} d\mu(y) = \int_E \frac{\partial f}{\partial x}(x,y) d\mu(y)$$

**Example VI.5:** *Differentiability of the Euler gamma function*

With the same notations as in the previous Example VI.3, since

$$\frac{\partial f}{\partial x}(x,y) = \ln(y) f(x,y)$$

and

$$\left|\frac{\partial f}{\partial x}(x,y)\right| = |\ln(y)|\, g(y)$$

with $|\ln|\, g$ integrable on $\mathbb{R}_+$ we have that $\forall x \in \mathbb{R}_+^*$

$$\Gamma'(x) = \int_{\mathbb{R}_+} \ln(y) y^{x-1} e^{-1} dy$$

**Theorem VI.6:** *Change of variable for the push-forward measure*

Let $f : (E, \mathcal{A}, \mu) \to (F, \beta)$ measurable, we define the push-forward measure of $\mu$ by $f$, $\forall M \in \mathcal{A}$ by

$$f_* \mu(M) \coloneqq \mu\left(f^{-1}(M)\right)$$

If $g : (F\beta) \to \overline{\mathbb{R}}$ is measurable then,

$$\int_F g d(f_* \mu) = \int_E g \circ f d\mu$$

Informally this is a change of variable if one defines $y \coloneqq f^{-1}(x), x = f(y)$ as

$$\text{"} \int_F g(x) d\mu\left(f^{-1}(x)\right) = \int_E g(f(y)) d\mu(y) \text{"}$$

**Proof:**

First, we verify that $f_* \mu$ is a measure as

$$f^{-1}(\varnothing) = \varnothing$$

and for $(M_n)_{n\in\mathbb{N}} \subseteq \beta$ disjoints,

$$f^{-1}\left(\bigsqcup_{n\in\mathbb{N}} M_n\right) = \bigsqcup_{n\in\mathbb{N}} f^{-1}(M_n)$$

Let $M \in \beta$, noticing that $\mathbb{1}_{f^{-1}(M)} = \mathbb{1}_M \circ f$, we obtain the desired formula for an indicator function:

$$\int_F \mathbb{1}_M df_*\mu = f_*\mu(M) = \mu\left(f^{-1}(M)\right) = \int_E \mathbb{1}_{f^{-1}(M)} d\mu = \int_E \mathbb{1}_M \circ f d\mu$$

the result is then generalized to measurable functions by linearity and monotone convergence.

**Remark VI.7:** *Link with probability theory*

A measured space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space if $\mathbb{P}(\Omega) = 1$. In this case we say that

- $\Omega$ is the sample space (the space of all outcomes)
- $\mathcal{F}$ is the set of all events
- $\mathbb{P}$ is a probability

Let $\mathcal{A} \in \mathcal{F}$, $\mathbb{P}(A)$ is the probability of the event $A \subseteq \Omega$.
A random real variable on $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable function $X : \Omega \to \mathbb{R}$. The law of $X$ is then $\mathbb{P}_X \coloneqq X_*\mathbb{P}$. It is a probability on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as

$$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}\left(X^{-1}(\mathbb{R})\right) = \mathbb{P}(\Omega) = 1$$

Given an event $E \in \mathcal{B}(\mathbb{R})$, in probability, we usually denote $(X \in E) \coloneqq X^{-1}(E)$ so that

$$\mathbb{P}(X \in E) = \mathbb{P}_X(E) = \mathbb{P}(X^{-1}(E))$$

Let $g$ be a real measurable function, then the push-forward formula provides an expression for the computation of the expectation

$$\mathbb{E}[g \circ X] \coloneqq \int_\Omega g \circ X d\mathbb{P} = \int_\mathbb{R} g(x) d\mathbb{P}_X(x)$$

which is convenient to compute expectations as a real integral.
We say that the law $\mathbb{P}_X$ has a density $f_X : \mathbb{R} \to \mathbb{R}_+$ if

$$\forall E \in \mathbb{B}(\mathbb{R}), \mathbb{P}_X(E) = \int_E f_X(x) dx$$

and in this case expectations can be computed with real integral with respect to the Lebesgue measure since

$$\mathbb{E}[g \circ X] = \int_\mathbb{R} g(x) f(x) dx$$

Indeed, this follows from the fact that $\forall E \in \mathbb{B}(\mathbb{R})$,

$$\int_{\mathbb{R}} \mathbb{1}_E(x) d\mathbb{P}_X(x) = d\mathbb{P}_X(E) = \int_E f_X(x) dx = \int_{\mathbb{R}} \mathbb{1}_E(x) f(x) dx$$

# VII Annexes

## VII.1 Completion of measures

1. Prove that $\overline{\mathcal{A}}^{\mu} \subseteq \sigma(\mathcal{A} \cup \mathcal{N}_{\mu})$.
2. Prove that $\mathcal{A} \cup \mathcal{N}_{\mu} \subseteq \overline{\mathcal{A}}^{\mu}$.
3. Prove that $\overline{\mathcal{A}}^{\mu}$ is a $\sigma$-Algebra over $E$.
4. Conclude that $\sigma(\mathcal{A} \cup \mathcal{N}_{\mu}) = \overline{\mathcal{A}}^{\mu}$.

Let $M \in \sigma(\mathcal{A} \cup \mathcal{N}_{\mu})$, we choose $\exists N_1, N_2 \in \mathcal{A} | N_1 \subseteq M \subseteq N_2$ and $\mu(N_2 \backslash N_1) = 0$ according to the definition of $\overline{\mathcal{A}}^{\mu}$.

5. Verify that $\mu(N_1) = \mu(N_2)$.

We define

$$\overline{\mu}(M) \coloneqq \mu(N_1) = \mu(N_2).$$

6. Verify that $\overline{\mu}$ is well defined, i.e. that its definition is independent of the choice of measurable sets $N_1, N_2$.
7. Prove that $\overline{\mu}$ is a measure on $\left(E, \overline{\mathcal{A}}^{\mu}\right)$.
8. Prove that $\overline{\mu}$ is the only measure extending $\mu$ on $\left(E, \overline{\mathcal{A}}^{\mu}\right)$.
9. Verify that the measured space $\left(E, \overline{\mathcal{A}}^{\mu}, \overline{\mu}\right)$ is complete, i.e. contains all the $\overline{\mu}$-negligible sets.
10. Prove that $\left(E, \overline{\mathcal{A}}^{\mu}, \overline{\mu}\right)$ is the smallest complete extension of $(E, \mathcal{A}, \mu)$. Precisely, you need to verify that if $(E, \mathcal{B}, \eta)$ is a complete measured space such that $\mathcal{A} \subseteq \mathcal{B}$ and $\eta_{|\mathcal{A}} = \mu$ then $\overline{\mathcal{A}}^{\mu} \subseteq \mathcal{B}$.

## VII.2 From outer measures to measures

Let $A, B \in \mathcal{A}_{\mu}^{*}$ and $N \subseteq \mathcal{P}(E)$.

1. Prove that

$$\mu^*(N) = \mu^*(N \cap A \cap B) + \mu^*(N \cap A \cap B^c) + \mu^*(N \cap A^c \cap B) + \mu^*(N \cap A^c \cap B^c)$$

2. Apply this to $N \cap (A \cup B)$ instead of $N$.
3. Deduce that $\mathcal{A}_{\mu}^{*}$ is closed under union.

4. Prove that if $A \cap B = \varnothing$ then

$$\mu^*(N \cap (A \cup B)) = \mu^*(N \cap A) + \mu^*(N \cap B)$$

5. Generalize the previous result by induction.
6. Prove that $\mathcal{A}_\mu^*$ is non-empty and closed under complement.

Let $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_\mu^*$

7. Construct a two by two disjoint sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_\mu^*$ such that

$$\forall n \in \mathbb{N}, \bigcup_{i=0}^{n} A_i = \bigcup_{i=0}^{n} M_i$$

8. Prove that $\forall n \in \mathbb{N}$

$$\mu^*(N) \geqslant \sum_{i=0}^{n} \mu^*(N \cap A_i) + \mu^*\left(N \cap \left(\bigcup_{i\in\mathbb{N}} M_i\right)^c\right)$$

9. Prove that

$$\mu^*(N) \geqslant \mu^*\left(N \cap \bigcup_{n\in\mathbb{N}} M_n\right) + \mu^*\left(N \cap \left(\bigcup_{n\in\mathbb{N}} M_n\right)^c\right) \tag{VII.1}$$

10. Conclude that $\mathcal{A}_\mu^*$ is a $\sigma$-algebra

Now, assume that $(M_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_\mu^*$ is two by two disjoint.

11. Prove that $\mu^*_{|\mathcal{A}_\mu^*}$ is $\sigma$-additive by choosing

$$N := \bigcup_{n\in\mathbb{N}} M_n$$

inside (VII.1).

12. Conclude that $\mu^*_{|\mathcal{A}_\mu^*}$ is a measure.
13. Prove that $\mathcal{A}_\mu^*$ is complete.

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