Mean-Field limit of the Bose-Hubbard model in high dimension

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August 12th 2026 solar eclipse from the Prarion summit.

[1] S.Farhat D.Périce S.Petrat. "Mean-Field Dynamics of the Bose-Hubbard Model in High Dimension". In: *Annales Henri Poincaré* (to appear).

[2] S.Farhat D.Périce S.Petrat. "Ground state energy of the Bose–Hubbard model with large coordination number with a polaron-type quantum de Finetti theorem". In: *arXiv:2603.29562* (2026).

Convergence of the ground state energy

$d-$dimensional cubic lattice: of length $L \in \mathbb{N}^*$: $\Lambda_d \coloneqq (\mathbb{Z}/L\mathbb{Z})^d$.

Bose–Hubbard Hamiltonian acting on $\mathcal{F}_d \coloneqq \ell^2(\mathbb{C})^{\otimes|\Lambda_d|}$:

$$H_d \coloneqq -\frac{J}{2d} \overbrace{\sum_{\substack{x,y\in\Lambda_d \\ |x-y|=1}} a_x^\dagger a_y}^{\mathcal{O}(2d|\Lambda_d|)} + (J-\mu)\sum_{x\in\Lambda_d} \mathcal{N}_x + \frac{U}{2}\sum_{x\in\Lambda_d} \mathcal{N}_x(\mathcal{N}_x - 1).$$

Mean-field Hamiltonian given $\varphi \in \ell^2(\mathbb{C})$:

$$h_\varphi \coloneqq -J\left(\overline{\alpha_\varphi} a + \alpha_\varphi a^\dagger - |\alpha_\varphi|^2\right) + (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1),$$

of order parameter $\alpha_\varphi \coloneqq \langle \varphi, a\varphi \rangle$.

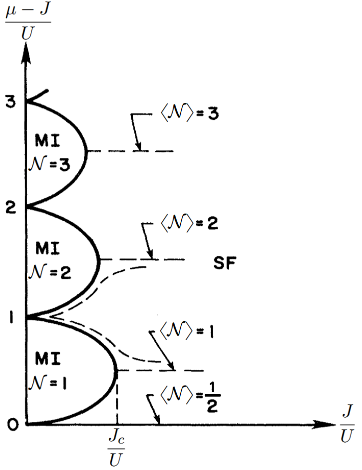
Theorem: *S.Farhat & D.P. & S.Petrat 2026 [2]*

If $U > 0$ and $J \geqslant 0$, then

$$\inf_{\psi\in\mathcal{F}_d,\ \|\psi\|_{\mathcal{F}_d}=1} \frac{\langle \psi, H_d\psi\rangle}{|\Lambda_d|} \underset{d\to\infty}{\longrightarrow} \inf_{\varphi\in\ell^2(\mathbb{C}),\ \|\varphi\|_{\ell^2}=1} \langle \varphi, h_\varphi \varphi\rangle.$$

Phase transition: for a mean-field energy minimizer φ_0,

- Mott Insulator: $\langle \varphi_0, a\varphi_0\rangle = 0$.
- Superfluid: $\langle \varphi_0, a\varphi_0\rangle > 0$.



Convergence of the dynamics

Schrödinger equation: Let $\Psi_d \in \mathcal{F}_d$ solve

$$i\partial_t \Psi_d(t) = H_d \Psi_d(t), \qquad H_d \coloneqq -\frac{J}{2d} \sum_{\substack{x,y\in\Lambda_d \\ |x-y|=1}} a_x^\dagger a_y + (J-\mu)\sum_{x\in\Lambda_d} \mathcal{N}_x + \frac{U}{2}\sum_{x\in\Lambda_d} \mathcal{N}_x(\mathcal{N}_x - 1). \tag{BH}$$

Mean-field dynamics: Let $\varphi \in \ell^2(\mathbb{C})$ solve

$$i\partial_t \varphi(t) = h_{\varphi(t)}\varphi(t), \qquad h_\varphi \coloneqq -J(\overline{\alpha_\varphi} a + \alpha_\varphi a^\dagger - |\alpha_\varphi|^2) + (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1), \qquad \alpha_\varphi = \langle \varphi, a\varphi\rangle. \tag{MF}$$

Reduced density matrix: Let $\gamma_d \coloneqq |\Psi_d\rangle\langle\Psi_d|$ and define the one-lattice site reduced density matrix

$$\gamma_d^{(1)} \coloneqq \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \mathrm{Tr}_{\Lambda\setminus\{x\}}(\gamma_d), \qquad p_\varphi \coloneqq |\varphi\rangle\langle\varphi|, \qquad q_\varphi \coloneqq 1 - p_\varphi.$$

Theorem: *S.F. & D. Périce & S. Petrat 2025 [1]*

We assume:

- Ψ_d solves (BH) with $\|\Psi_d(0)\|_{\mathcal{F}_d} = 1$,
- φ solves (MF) with $\|\varphi(0)\|_{\ell^2} = 1$,
- $U > 0$ and there is $C > 0$ such that

$$\mathrm{Tr}(p_\varphi(0)\mathcal{N}^4) \leqslant C.$$

Then, there exists $C(J,\mu,U) > 0$ such that for all $t \in \mathbb{R}_+$,

$$\|\gamma_d^{(1)}(t) - p_\varphi(t)\|_{\mathcal{L}^1} \leqslant C(J,\mu,U) e^{C(J,\mu,U)(1+t^7)} \left(\mathrm{Tr}\left[\gamma_d^{(1)}(0)\left(q_\varphi(0) + q_\varphi(0)\mathcal{N}^2 q_\varphi(0)\right)\right] + \frac{1}{d}\right)^{1/2}.$$