Ground state of the Bose–Hubbard model with large coordination number

Graphs: $(V_z, E_z)_{z\in\mathbb{N}*}$ with homogeneous coordination number z.

Bose–Hubbard Hamiltonian:

$$H_z := -\frac{J}{z}\overbrace{\sum_{\{x,y\}\in E_z}\left(a_x^\dagger a_y + a_y^\dagger a_x\right)}^{\mathcal{O}(z|V_z|)} + (J-\mu)\sum_{x\in V_z}\mathcal{N}_x + \frac{U}{2}\sum_{x\in V_z}\mathcal{N}_x(\mathcal{N}_x-1),$$

acting on $\mathcal{F}_z := \ell^2(\mathbb{C})^{\otimes|V_z|}$.

Mean-field Hamiltonian given $\varphi \in \ell^2(\mathbb{C})$:

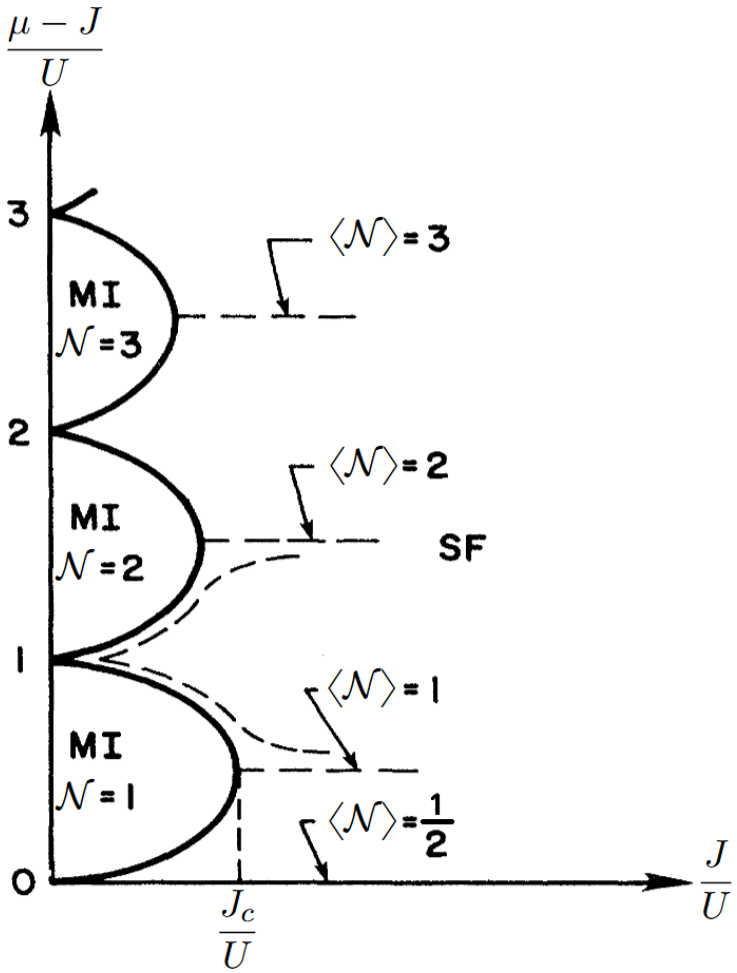
$$h_\varphi := -J\left(\overline{\alpha_\varphi}a + \alpha_\varphi a^\dagger - |\alpha_\varphi|^2\right) + (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1),$$

of order parameter $\alpha_\varphi := \langle\varphi, a\varphi\rangle$.

Theorem: *Convergence of the ground state energy*

If $U > 0$ and $J \geqslant 0$, then

$$\inf_{\psi\in\mathcal{F}_z,\ \|\psi\|_{\mathcal{F}_z}=1}\frac{\langle\psi, H_z\psi\rangle}{|V_z|} \underset{z\to\infty}{\longrightarrow} \inf_{\varphi\in\ell^2(\mathbb{C}),\ \|\varphi\|_{\ell^2}=1}\langle\varphi, h_\varphi\varphi\rangle.$$



Phase transition: for a mean-field energy minimizer φ_0,

- Mott Insulator: $\langle\varphi_0, a\varphi_0\rangle = 0$.
- Superfluid: $\langle\varphi_0, a\varphi_0\rangle > 0$.