Mean field limits for 2D fermions under large magnetic field

Denis Périce
PHD supervisor : Nicolas Rougerie

UMPA

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Problem definition

Model :

We consider spinless fermions in a 2D plane with perpendicular uniform magnetic field in a confining potential.

Mean field scaling Hamiltonian :

$$H_N := \sum_{j=1}^{N} \left(\left[-i\nabla_j - \frac{B}{2} x_j^{\perp} \right]^2 + N V_j \right) + \sum_{1 \leq i < j \leq N} w_{ij} \tag{1}$$

acting on $L^2_{asym}(\mathbb{R}^{2N}) := \bigwedge^N L^2(\mathbb{R}^2)$

Large magnetic field limit :

$$N \to \infty, B \to \infty, \frac{B}{N} \to \infty \tag{2}$$

Approximate energy functional

$$\mathcal{E}_{class}[\rho] = \int_{\mathbb{R}^2} V\rho + \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(x-y)\rho(x)\rho(y)dxdy \tag{3}$$

Definitions :

- $E_N^0 := \inf\left\{\langle\Psi_N|H_N|\Psi_N\rangle, \Psi_N \in L^2_{asym}(\mathbb{R}^{2N}), \langle\Psi_N|\Psi_N\rangle = 1\right\}$
- $\mathcal{E}^0_{class} := \inf\left\{\mathcal{E}_{class}[\rho], \rho \geq 0, \int_{\mathbb{R}^2}\rho = 1\right\}$

Theorem 1 : Convergence in large magnetic field limit

If $V(x) \underset{|x|\to\infty}{\to} \infty$, $w \geq 0$, and some regularities assumptions on potentials V and w, we have in the large magnetic field limit :

$$E(N) := \frac{E_N^0 - NB}{N^2} \to \mathcal{E}^0_{class} \tag{4}$$

Mean field scaling

Characteristic lengths :

- $N^{-\frac{1}{2}}$ for particle density
- $l_B := \frac{1}{\sqrt{B}}$, the magnetic length

The square ratio is $\frac{\frac{1}{N}}{l_B^2} = \frac{B}{N}$

Other scaling with $\tilde{B} := \frac{B}{\sqrt{N}}$ and $\hbar = N^{-1/2}$:

$$\frac{H_N}{N} = \sum_{j=1}^{N} \left(\left[-i\hbar\nabla_j - \frac{\tilde{B}}{2} x_j^{\perp} \right]^2 + V_j \right) + \frac{1}{N} \sum_{1 \leq i < j \leq N} w_{ij} \tag{5}$$

Landau levels

We focus on the kinetic part of the Hamiltonian :

$$H_N^0 = \sum_{j=1}^{N} \left(\left[-i\nabla_j - \frac{B}{2} x_j^{\perp} \right]^2 \right) = \sum_{j=1}^{N} \begin{pmatrix} \pi_x^j \\ \pi_y^j \end{pmatrix}^2 \tag{6}$$

Classical phase space : $(n, R) \in \mathbb{N} \times \mathbb{R}^2$
➔ we decompose the position operator : $r = R + \tilde{R}$, with :

$$\tilde{R} := l_B^2 \begin{pmatrix} -\pi_y \\ \pi_x \end{pmatrix} \tag{7}$$

Cyclotron orbit quantization

- $\tilde{R}$ represents the cyclotron orbit part
- For one particle, we have the Landau level quantization :

$$H_1^0 = 2B\left(a^\dagger a + \frac{1}{2}\right) \tag{8}$$

with :

$$a^\dagger = \frac{\tilde{R}_x - i\tilde{R}_y}{\sqrt{2}l_B} \quad a = \frac{\tilde{R}_x + i\tilde{R}_y}{\sqrt{2}l_B} \quad \text{satisfying} \quad [a, a^\dagger] = \mathbb{1} \tag{9}$$

Guiding center quantization

R represents the guiding center of the orbit :

$$b^\dagger = \frac{R_x + iR_y}{\sqrt{2}l_B} \quad b = \frac{R_x - iR_y}{\sqrt{2}l_B} \quad \text{satisfying} \quad [b, b^\dagger] = \mathbb{1} \tag{10}$$

We have the Hilbert basis :

$$\varphi_{nm} = \frac{{a^\dagger}^n {b^\dagger}^m}{\sqrt{n!m!}}\varphi_{00} = \frac{{b^\dagger}^m {a^\dagger}^n}{\sqrt{n!m!}}\varphi_{00} \quad \text{where } \varphi_{00} \text{ is Gaussian} \tag{11}$$

We can now define the projectors :

- projector on nLL : $\Pi_n := \sum_{m=0}^{\infty} |\varphi_{nm}\rangle \langle\varphi_{nm}|$
- space localisation : $\Pi_{n,R}(x,y) = g(x-R)\Pi_n(x-y)g(y-R)$

Resolution of identity : $\sum_{n=0}^{\infty} \Pi_n = \mathbb{1}$ and $\sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \Pi_{n,R} = \mathbb{1}$

Energy functional

Let Γ_N be a density matrix on $L^2_{asym}(\mathbb{R}^{2N})$, with $\gamma_N^{(1)}$ and $\gamma_N^{(2)}$ its first and second reduced densities.

The energy is :

$$\mathcal{E}_N[\Gamma_N] := \mathrm{Tr}(h\gamma_N^{(1)}) + \frac{1}{2}\mathrm{Tr}(w\gamma_N^{(2)}) \tag{12}$$

We define the Husimi functions :

- $m^{(1)}(n, R) := \mathrm{Tr}(\Pi_{n,R}\gamma_N^{(1)})$
- $m^{(2)}(n_1, n_2; R_1, R_2) := \mathrm{Tr}\left((\Pi_{n_1,R_1} \otimes \Pi_{n_2,R_2})\gamma_N^{(2)}\right)$

The total density is $\rho(x) := \gamma_N^{(1)}(x, x)$ andc

$$\rho = \sum_{n=0}^{\infty} m^{(1)}(n, .) + \text{error term} \tag{13}$$

$$
\begin{aligned}
\mathcal{E}_N[\Gamma_N] = &\sum_{n=0}^{\infty} 2B\left(n+\frac{1}{2}\right)\int_{\mathbb{R}^2} m^{(1)}(n,x)dx \\
&+ N\sum_{n=0}^{\infty}\int_{\mathbb{R}^2} V(x)m^{(1)}(n,x)dx \\
&+ \frac{1}{2}\sum_{n_1,n_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(x-y)m^{(2)}(n_1,x;n_2,y)dxdy + \text{error terms}
\end{aligned}
\tag{14}
$$

- We make the mean field approximation : $m^{(2)} = m^{(1)} \otimes m^{(1)}$
- m satisfy the Pauli principle : $0 \leq \Gamma_N \leq \mathbb{1} \implies 0 \leq m^{(1)}(n,R) \leq \frac{B}{2\pi}$
- By subtracting LLL energy, in the large magnetic field limit :

$$
\mathcal{E}_{class}[\rho] = \int_{\mathbb{R}^2} V\rho + \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(x-y)\rho(x)\rho(y)dxdy + \text{error terms}
\tag{15}
$$

Useful tools

Theorem 2 : (De Finetti or Hewitt-Savage)

Let $\mu \in \mathcal{P}_s(\Omega^{\mathbb{N}})$, a symmetric probability measure, there exist a probability measure $P_\mu \in \mathcal{P}(\mathcal{P}(\Omega))$ such that :

$$\forall n \in \mathbb{N}, \mu^{(n)} = \int_{\mathcal{P}(\Omega)} \rho^{\otimes n} dP_\mu(\rho) \tag{16}$$

where $\mu^{(n)}$ is the n^{th} marginal of μ

Theorem 3 : Lieb's variational principle

If γ_1 is a positive operator of trace N such that $\gamma_1 \leq \mathbb{1}$, then :

$$E_N^0 \leq \mathcal{E}_{HF}(\gamma_1) \tag{17}$$

with :

- $\gamma_2(z, w; z', w') := \gamma_1(z, z')\gamma_1(w, w') - \gamma_1(z; w')\gamma_1(w; z')$
- $\mathcal{E}_{HF}(\gamma_1) := \mathrm{Tr}(h\gamma_1) + \frac{1}{2}\mathrm{Tr}(w\gamma_2)$

Magnetic periodic boundary conditions

Goal :

- q lowest Landau levels fully filled
- q^{th} Landau level partially filled with ratio r

$\Omega = [0, L]^2$, take $B = \nabla \wedge A$ so $T_R A - A = \nabla \varphi_R$, and define :

$$\tau_R := e^{-i\varphi_R} T_R \tag{18}$$

Properties

- $[\tau_R, (i\nabla + A)^2] = 0$
- $\int_{\partial\Omega} A \cdot dl = BL^2 = 2\pi d$ where d is the is the degeneracy of the landau levels

➔ we fix, in the limit $\frac{N}{d} = \frac{2\pi N}{BL^2} = q + r$

Energy functional :

$$\mathcal{E}_{class}[m(q,\cdot)] = \int_\Omega Vm(q,\cdot) + \frac{1}{2}\int_\Omega\int_\Omega w(x-y)m(q,x)m(q,y)dxdy \tag{19}$$

with $\int m(q,x)dx = \frac{rd}{N} = \frac{r}{d+r}$ and $m(q,x) \leq \frac{1}{L^2(q+r)}$

We have the Hilbert basis :

$$\varphi_{nl} = \frac{1}{\sqrt{n!}} {a^\dagger}^n \tau^l_{-i\frac{L}{d}} \varphi_{00} = \frac{1}{\sqrt{n!}} \tau^l_{-i\frac{L}{d}} {a^\dagger}^n \varphi_{00} \tag{20}$$

where φ_{00} is a theta function :

$$|\varphi_{00}(z)| = c\left|\theta\left(\frac{d}{L}z, id\right)\right| \tag{21}$$

with $\theta(z,\tau) = \sum_{k\in\mathbb{Z}} e^{i\pi\tau k^2 + 2i\pi kz}$

Theorem 4 : Convergence with magnetic periodic conditions

If $w \geq 0$, and some regularities assumptions on potentials V and w, we have in limit $\frac{N}{d} = \frac{2\pi N}{BL^2} = q + r$:

$$E(N) := \frac{E_N^0 - E_q}{N^2} \to \mathcal{E}^0_{class} \tag{22}$$

Where E_q is the energy of q lowest Landau levels and their interactions with the $q^{th} level$.

Main steps in the proof of theorem 4

Upper bound :

- With $m(q,\cdot)$ minimizing $\mathcal{E}_{class}$, we build the test state :

$$\gamma_1 := \frac{2\pi N}{B} \int_{\Omega} m(q,R)\Pi_{q,R} dR \tag{23}$$

- With Lieb's result :

$$\lim E(N) \leq \mathcal{E}_{class}(\rho_{\gamma_1}) \quad \text{where} \quad \rho_{\gamma_1}(x) = \gamma_1(x,x) \tag{24}$$

- Varying g and using the following lemma :

$$\text{Uniformly,} \quad \Pi_{n,L}(z,z) \sim \Pi_n(z,z) = \frac{B}{2\pi} \tag{25}$$

we show :

$$\lim E(N) \leq \mathcal{E}^0_{class} \tag{26}$$

Lower bound :

Let $(\Gamma_N)_{N\in\mathbb{N}}$ be a minimizing sequence of $\lim E(N)$

- Extract a weakly* convergent sequence from $m_N^{(2)}(q,q;\cdot)$
- With Fatou inequality :

$$\liminf \int_\Omega \int_\Omega [w(x-y)+V(x)+V(y)]\, dm_N^{(2)}(x,y) \geq \int_\Omega \int_\Omega [w(x-y)+V(x)+V(y)]\, dm^{(2)}(x,y) \tag{27}$$

- Then, with De Finetti theorem $m^{(2)} = \int_{\mathcal{P}(\Omega)} m^{\otimes 2} dP_\mu(m)$, so :

$$\lim E(N) \geq \frac{1}{2}\int_{\mathcal{P}(\Omega)} \int_\Omega \int_\Omega [w(x-y)+V(x)+V(y)]\, dm^{\otimes 2}(x,y) dP_\mu(m) \geq \mathcal{E}^0_{class} \tag{28}$$

Prospects

- Generalize to the case of non repulsing interactions
  ➔ we must construct a Slater determinant as test state instead of using Lieb's theorem
- Study some related evolution problems

Thank for your attention :)

References

S. Fournais, M. Lewin, and J-P. Solovel.
The semi-classical limit of large fermionic systems.
arXiv:1510.01124, 2015.

S. Fournais and P. Madsen.
Semi-classical limit of confined fermionic systems in homogeneous magnetic fields.
arXiv:1907.00629, 2019.

E. H. Lieb, J-P. Solovej, and J. Yngvason.
Ground states of large quantum dots in magnetic fields.
Phys. Rev. B, 51:10646–10665, Apr 1995.

E. H. Lieb and R. Seiringer.
The stability of matter in quantum mechanics.
2008.

N. Rougerie and J. Yngvason.
Holomorphic quantum hall states in higher landau levels.
arXiv:1912.10904, 2019.

N. Rougerie
Théorèmes de De Finetti, limites de champ moyen et condensation de Bose-Einstein.
hal-01060125v4, 2014.

E. H. Lieb J-P. Solovej
Quantum Dots
arXiv:cond-mat/9404099v1, 1994.

E. H. Lieb
Variational Principle for Many-Fermion Systems
Phys. Rev. Volume 46, 1981, Number 7

T. Champel and S. Florens.
Quantum transport properties of two-dimensional electron gases under high magnetic fields
Phys. Rev. B, 75 (2007), p. 245326.

Eugene A. Feinberg, Pavlo O. Kasyanov, Nina V. Zadoianchuk
T. Champel and S. Florens.
Fatou's Lemma for Weakly Converging Probabilities
arXiv:1206.4073v3, 2012.

Y. Almog
Abrikosov lattices in finite domain
Commun. Math. Phys. (2005), DOI : 10.1007/s00220-005-1463-x