Mean field limit for 2D fermions in large magnetic field

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Problem definition

Model :

We consider spinless fermions in a 2D plane with perpendicular uniform magnetic field in a confining potential.

Mean field scaling Hamiltonian :

$$H_N := \sum_{j=1}^{N} \left(\left[-i\nabla_j - \frac{B}{2} x_j^{\perp} \right]^2 + N V_j \right) + \sum_{1 \le i < j \le N} w_{ij} \tag{1}$$

acting on $L^2_{asym}(\mathbb{R}^{2N}) := \bigwedge^N L^2(\mathbb{R}^2)$

Large magnetic field limit :

$$N \to \infty, B \to \infty, \frac{B}{N} \to \infty \tag{2}$$

Approximate energy functional

$$\mathcal{E}_{class}[\rho] = \int_{\mathbb{R}^2} V\rho + \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)w(x-y)\rho(y)dxdy \qquad (3)$$

we define :

- $E_N^0 := \inf\left\{\langle\Psi_N|H_N|\Psi_N\rangle, \Psi_N \in L^2_{asym}(\mathbb{R}^{2N}), \langle\Psi_N|\Psi_N\rangle = 1\right\}$
- $\mathcal{E}^0_{class} := \inf\left\{\mathcal{E}_{class}[\rho], \rho \geq 0, \int_{\mathbb{R}^2}\rho = 1\right\}$

Theorem 1 : Convergence in large magnetic field limit

If $V(x) \underset{|x|\to\infty}{\to} \infty$, $w \geq 0$, and some regularities assumptions on potentials V and w, we have in the large magnetic field limit :

$$E(N) := \frac{E_N^0 - NB}{N^2} \to \mathcal{E}^0_{class} \qquad (4)$$

Mean field scaling

Characteristic lengths :

- $N^{-\frac{1}{2}}$, linked to the particle density
- l_B, the magnetic length, defined by : $l_B^2 = B^{-1}$

The square ratio is $\frac{\frac{1}{N}}{l_B^2} = \frac{B}{N}$

Example :

Model for a neutral atom of atomic number $Z = N$:

$$H = \sum_{i=1}^{N} \left(-\Delta_i - \frac{Z}{|x_i|} \right) + \sum_{1 \leq i < j \leq N} \frac{1}{|x_j - x_i|} \tag{5}$$

Other scaling with $\tilde{B} := \frac{B}{\sqrt{N}}$ and $\hbar = N^{-1/2}$:

$$\frac{H_N}{N} = \sum_{j=1}^{N} \left(\left[-i\hbar\nabla_j - \frac{\tilde{B}}{2} x_j^{\perp} \right]^2 + V_j \right) + \frac{1}{N} \sum_{1 \leq i < j \leq N} w_{ij} \tag{6}$$

Landau levels

We focus on the kinetic part of the Hamiltonian :

$$H_N^0 = \sum_{j=1}^{N} \left(\left[-i\nabla_j - \frac{B}{2} x_j^{\perp} \right]^2 \right) = \sum_{j=1}^{N} \begin{pmatrix} \pi_x^j \\ \pi_y^j \end{pmatrix}^2 \qquad (7)$$

Classical phase space :

$(n, R) \in \mathbb{N} \times \mathbb{R}^2$

➔ we decompose the position operator : $r = R + \tilde{R}$, with :

$$\tilde{R} := \frac{l_B^2}{\pi} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (8)$$

Cyclotron orbit quantization

- $\tilde{R}$ represents the cyclotron orbit part
- For one particle, we have the Landau level quantization :

$$H_1^0 = 2B\left(a^\dagger a + \frac{1}{2}\right) \tag{9}$$

with :

$$a^\dagger = \frac{\tilde{R}_x - i\tilde{R}_y}{\sqrt{2}l_B} \quad a = \frac{\tilde{R}_x + i\tilde{R}_y}{\sqrt{2}l_B} \quad \text{satisfying} \quad [a, a^\dagger] = \mathbb{1} \tag{10}$$

Guiding center quantization

R represents the guiding center of the orbit :

$$b^\dagger = \frac{R_x + iR_y}{\sqrt{2}l_B} \qquad b = \frac{R_x - iR_y}{\sqrt{2}l_B} \qquad \text{satisfying} \qquad [b, b^\dagger] = \mathbb{1} \qquad (11)$$

We have the Hilbert basis :

$$\varphi_{nm} = \frac{{a^\dagger}^n {b^\dagger}^m}{\sqrt{n!m!}}\varphi_{00} = \frac{{b^\dagger}^m {a^\dagger}^n}{\sqrt{n!m!}}\varphi_{00} \qquad \text{where } \varphi_{00} \text{ is Gaussian} \qquad (12)$$

We can now define the projectors :

- projector on nLL : $\Pi_n := \sum_{m=0}^{\infty} |\varphi_{nm}\rangle \langle\varphi_{nm}|$
- space localisation : $\Pi_{n,R}(x,y) = g(x-R)\Pi_n(x-y)g(y-R)$

Resolution of identity : $\sum_{n=0}^{\infty} \Pi_n = \mathbb{1}$ and $\sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \Pi_{n,R} = \mathbb{1}$

Energy functional

Let Γ_N be a density matrix on $L^2_{asym}(\mathbb{R}^{2N})$, with $\gamma_N^{(1)}$ and $\gamma_N^{(2)}$ its first and second reduced densities.

The energy is :

$$\mathcal{E}_N[\Gamma] := \mathrm{Tr}(h\gamma_N^{(1)}) + \frac{1}{2}\mathrm{Tr}(w\gamma_N^{(2)}) \tag{13}$$

We define the Husimi functions :

- $m^{(1)}(n, R) := \mathrm{Tr}(\Pi_{n,R}\gamma_N^{(1)})$
- $m^{(2)}(n_1, R_1; n_2, R_2) := \mathrm{Tr}\left((\Pi_{n_1,R_1} \otimes \Pi_{n_2,R_2})\gamma_N^{(2)}\right)$

with $\rho(x) = \gamma_N^{(1)}(x, x)$, we have

$$\rho = \sum_{n=0}^{\infty} m^{(1)}(n, .) + \text{error term} \tag{14}$$

$$\mathcal{E}_N[\Gamma_N] = \sum_{n=0}^{\infty} 2B\left(n+\frac{1}{2}\right)\int_{\mathbb{R}^2} m^{(1)}(n,x)dx + N\sum_{n=0}^{\infty}\int_{\mathbb{R}^2} V(x)m^{(1)}(n,x)dx + \frac{1}{2}\sum_{n_1,n_2}\int_{\mathbb{R}^2\times\mathbb{R}^2} w(x-y)m^{(2)}(n_1,x;n_2,y)dxdy + \text{error terms} \tag{15}$$

- We make the mean field approximation : $m^{(2)} = m^{(1)} \otimes m^{(1)}$
- Bring back quantum aspects with semi-classical approximation
  ➔ m satisfy the Pauli principle $0 \leq m^{(1)}(n,R) \leq B$
- By subtracting LLL energy, in the large magnetic field limit :

$$\mathcal{E}_{class}[\rho] = \int_{\mathbb{R}^2} V\rho + \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)w(x-y)\rho(y)dxdy + \text{error terms} \tag{16}$$

De Finetti Theorem

- Rigorous justification for mean field assumption
- A symmetric probability measure of many variables is almost sum of decorrelated probabilities

Theorem 3 : (Hewitt-Savage)

Let $\mu \in \mathcal{P}_s(\Omega^{\mathbb{N}})$, a symmetric probability measure, there exist a probability measure $P_\mu \in \mathcal{P}(\mathcal{P}(\Omega))$ such that :

$$\forall n \in \mathbb{N}, \mu^{(n)} = \int_{\rho \in \mathcal{P}(\Omega)} \rho^{\otimes n} dP_\mu(\rho) \tag{17}$$

where $\mu^{(n)}$ is the n^{th} marginal of μ

Lieb variational principle

- Let h and w be the one body and two body operators in H_N
- Let γ_1 be a one particle density matrix

We define :

- $\gamma_2(z, w; z', w') := \gamma_1(z, z')\gamma_1(w, w') - \gamma_1(z; w')\gamma_1(w; z')$
- $\mathcal{E}_{HF}(\gamma_1) := \mathrm{Tr}(h\gamma_1) + \frac{1}{2}\mathrm{Tr}(w\gamma_2)$

➔ This equations are satisfied if γ_1 and γ_2 are the reduced densities of a Slater determinant : $\mathcal{E}_N(\gamma_N^{(1)}) = \mathcal{E}_{HF}(\gamma_N^{(1)})$

Theorem 2 : Lieb's variational principle

If γ_1 is a positive operator of trace N such that $\gamma_1 \leq \mathbb{1}$, then :

$$E_N^0 \leq \mathcal{E}_{HF}(\gamma_1) \tag{18}$$

Main steps in the proof of theorem 1

Upper bound :

- With ρ_{class} minimizing $\mathcal{E}_{class}$, we build the test state :

$$\gamma_1 := \sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \tau(n,R)\Pi_{n,R} dR \text{ with } \tau(n,R) := \frac{N\rho_{class}(R)}{B}\delta_{0n} \tag{19}$$

where $\tau(n,R)$ is the filling factor : local density at R in nLL divided by maximum density

- With Lieb variational principle, using $w \geq 0$,

$$\lim E(N) \leq \mathcal{E}_{class}(\rho_{\gamma_1}) \quad \text{where} \quad \rho_{\gamma_1}(x) = \gamma_1(x,x) \tag{20}$$

- Varying g, $\mathcal{E}_{class}(\rho_{\gamma_1})$ can be made arbitrary close to $\mathcal{E}^0_{class}$, and therefore :

$$\lim E(N) \leq \mathcal{E}^0_{class} \tag{21}$$

Lower bound :

Let $(\Gamma_N)_{N\in\mathbb{N}}$ be a minimizing sequence of $\lim E(N)$

- Due to the confining assumption on potentials, we can extract a weakly* convergent sequence
- With a Fatou inequality :

$$\liminf \int_{\mathbb{R}^2\times\mathbb{R}^2} [w(x-y)+V(x)+V(y)]\, d\rho_N^{(2)}(x,y) \tag{22}$$
$$\geq \int_{\mathbb{R}^2\times\mathbb{R}^2} [w(x-y)+V(x)+V(y)]\, d\rho^{(2)}(x,y)$$

- Then, with De Finetti theorem $\rho_N^{(2)} = \int_{\rho\in\mathcal{P}(\mathbb{R}^2)} \rho^{\otimes 2} dP_\mu(\rho)$, so :

$$\lim E(N) \tag{23}$$
$$\geq \frac{1}{2}\int_{\rho\in\mathcal{P}(\mathbb{R}^2)}\int_{\mathbb{R}^2\times\mathbb{R}^2} [w(x-y)+V(x)+V(y)]\, d\rho^{\otimes 2}(x,y)dP_\mu(\rho) \tag{24}$$
$$\geq \mathcal{E}^0_{class}$$

Prospects

Explore a weaker magnetic field limit : $\lim \frac{B}{N} \in \mathbb{R}$ is finite

➔ Several Landau levels are partially filled

- With a strong confining : $V = \infty$ outside of a bounded domain
- With a weaker confining : $V(x) \underset{|x| \to \infty}{\to} \infty$

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