GYROKINETIC LIMIT OF THE 2D HARTREE EQUATION IN A LARGE MAGNETIC FIELD

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ABSTRACT. We study the dynamics of two-dimensional interacting fermions submitted to a homogeneous transverse magnetic field. We consider a large magnetic field regime, with the gap between Landau levels set to the same order as that of potential energy contributions. Within the mean-field approximation, i.e. starting from Hartree's equation for the first reduced density matrix, we derive a drift equation for the particle density. We use vortex coherent states and the associated Husimi function to define a semi-classical density almost satisfying the limiting equation. We then deduce convergence of the density of the true Hartree solution by a Dobrushin-type stability estimate for the limiting equation.

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1. Introduction

Motivated in particular by the physical context of the quantum Hall effect [30, 22] we study the dynamics of many interacting 2D fermions in a large perpendicular magnetic field. At the many-body level the set-up would be that of the N-body Schrödinger equation

$$\begin{aligned} i\hbar\partial_t \Psi_N &= H_N \Psi_N \\ H_N &= \sum_{j=1}^N \left\{ \left(i\hbar\nabla_j + \frac{b}{2} x_j^\perp \right)^2 + NV(x_j) \right\} + \sum_{j<k} w(x_j - x_k) \end{aligned} \tag{1.1}$$

for $\Psi_N(t) \in L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$ an antisymmetric many-body wave-function. The second and third terms of the Hamiltonian were chosen to each formally weigh $O(N^2)$ in the large N limit. The first term, because of the Pauli exclusion principle encoded in the wave-function's antisymmetry and the nature of the spectrum of the magnetic Laplacian, will weigh $\simeq \max(N^2, bN)$ (say for fixed $\hbar$). What we mean by a large magnetic field limit is a scaling where $b \gtrsim N \to \infty$ with fixed $\hbar$, with time possibly rescaled appropriately. This will result in a combined mean-field and semi-classical[1] limit, as usual for many-fermions systems. Indeed the Pauli principle will impose the occupancy of a large number of single-particle quantum states, the hallmark of semi-classical regimes. The most crucial feature of the large magnetic field regime is that the appropriate classical phase-space is *not* the position/momentum (x, p) space. This being perhaps the most novel aspect of the problem, we shall for now bypass the justification of the mean-field approximation to focus on semi-classics.

Hence we start from the mean-field approximation of the above. This means replacing Ψ_N e.g. by a Slater determinant of N orthogonal one-body wave-functions and consider the time evolution of the projector on the subspace thus spanned. More generally, and with the above scaling conventions, this leads to Hartree's equation

$$i\hbar\partial_t \gamma = \left[H_\gamma, \gamma \right] \tag{1.2}$$

with $0 \leq \gamma(t) \leq \mathbb{1}$ a trace-class operator on $L^2(\mathbb{R}^2)$, that one should think of as being related to Ψ_N by a partial trace

$$\gamma = \mathrm{Tr}_{2\to N} |\Psi_N\rangle\langle\Psi_N|.$$

The mean-field Hamiltonian H_γ is given as

$$H_\gamma = \left(i\hbar\nabla + \frac{b}{2} x^\perp \right)^2 + NV + w \star \rho_\gamma \tag{1.3}$$

where

$$\rho_\gamma(x) = \gamma(x, x)$$

is the density of γ, defined in terms of its' operator kernel. The Pauli principle (antisymmetry of Ψ_N) translates into the operator constraint [33, Chapter 3]

$$0 \leq \gamma(t) \leq \mathbb{1}$$

while the number of particles is set as

$$\mathrm{Tr}\gamma = N.$$

The limiting dynamics can be guessed by studying that of a classical particle of charge -1 in a transverse magnetic field of amplitude b and a force field F. Newton's fundamental equation of dynamics gives

$$Z''(t) = F(t, Z(t)) + bZ'(t)^\perp \tag{1.4}$$

[1]Observe the respective roles of $\hbar$ and b in the kinetic energy operator $\left(i\hbar\nabla + \frac{b}{2} x^\perp \right)^2$.

For a constant and homogeneous force field, the motion is split into a cyclotron orbit and a drift (of the orbit's center) term

$$Z(t) = \underbrace{\frac{|Z_c'(0)|}{b}\begin{pmatrix}\cos(bt)\\ \sin(bt)\end{pmatrix}}_{=:Z_c(t)} + \underbrace{\frac{F^\perp}{b}t}_{=:Z_d(t)} \tag{1.5}$$

where we assumed

$$Z_d(0) = (0,0)$$
$$Z_c(0) = \frac{|Z_c'(0)|}{b}(1,0)$$

The characteristic time for the cyclotron orbit is b^{-1}, that for the drift is of order b. This suggests, for $b \to \infty$, to observe the motion over a time scale of order b, and that the cyclotron motion will be averaged over in the limit equation, its' radius being given by

$$r_c := \frac{|Z_c'(0)|}{b}.$$

Then, if we assume a more general force field F, slowly varying on the scale of the cyclotron orbit, we should expect to leading order an effective equation

$$Z_d'(t) = \frac{F^\perp}{b}$$

for the motion of the orbit center. Following (1.3) we should set

$$F = \nabla\left(NV + w \star \rho\right)$$

with ρ the density of particles, which leads via the method of characteristics to a transport-type equation

$$\partial_t \rho + \nabla^\perp\left(NV + w \star \rho\right) \cdot \nabla \rho = 0. \tag{1.6}$$

Our goal is to rigorously connect (1.2) to (1.6) in the limit $b \sim N \to \infty$. At the classical level this is a gyrokinetic limit.

At the quantum level, the cyclotron radius gets quantized in multiples of $\sqrt{n}$ where $n \in \mathbb{N}$ corresponds to the Landau level index labelling the eigenstates of the magnetic Laplacian. I.e. we write the spectral decomposition of the latter as

$$\left(i\nabla + \frac{b}{2}x^\perp\right)^2 = \sum_{n\in N} 2b\left(n + \frac{1}{2}\right)\Pi_n$$

where Π_n is an orthogonal projection (see below for more details). The main difference between the large magnetic field limit we consider here and more common analysis over the (x, p) semi-classical phase-space is that the gap $2b$ between Landau levels will be of the same order as other energetic contributions. As a consequence our phase-space will be parameterised by $R, n \in \mathbb{R}^2 \times \mathbb{N}$, corresponding to the center of the cyclotron orbit's center and the Landau level index/quantized cyclotron motion.

Corresponding static problems have been considered at the level of energy ground states in the series of works [35, 36, 50] for large atoms (see also [19]) and, more related to our context, for quantum dots in [37] (see also [40]). In particular it is found that for $b \ll N$, the problem reduces to leading order to a (x, p) semi-classical one, similar to the weak magnetic field situation of [34, 49, 18]. The limit energy is the usual Vlasov/Thomas-Fermi functional. By contrast, for $b \gtrsim N$, one finds a magnetic Thomas-Fermi theory set on the (R, n) phase-space. This is the case under consideration here, seemingly for the first time at the dynamical level.

Indeed, dynamical studies of large fermionic systems we are aware of, even for non-zero magnetic fields, all proceed on the (x, p) phase-space. The classical counterpart of this work, i.e. the gyrokinetic limit of the Vlasov equation, has been well studied [26, 8, 20, 6, 28, 21, 38, 27, 47, 48, 7]. Some results start from Newton's dynamics [29]. In the quantum literature it is known that the Hartree equation can be obtained by a mean field limit from the N-body Schrödinger dynamics [5, 25, 24, 41]. It is also known that the Vlasov equation can be derived in a semi-classical limit from the Hartree equation [1, 2, 4, 32, 31, 46]. The mean field and semi-classical limits can be coupled to obtain directly the Vlasov equation from the N-body Schrödinger dynamics [12, 13, 14]. More recent results have been dealing with singular potentials [45, 42, 15, 16].

Closer to our setting, we mention the recent [3] where Euler's equation in vorticity form is obtained from the N-body Schrödinger dynamics with large magnetic field and repulsive 2D Coulomb interaction $w = -\log|\,.\,|$. This corresponds to the drift equation (1.6) in this context. The crucial difference between [3] and the present contribution is that the former is set in a regime where the gap between Landau levels is small compared to the interactions. The classical phase-space is consequently again the position/momentum one. However, [3] can deal with a much more singular interaction potential, leveraging its' coercivity.

As regards the approach to the semi-classical limit, in the (x, p) phase-space, the use of the Wigner function is often the privileged angle of attack. We do not see that such a tool is available for the (R, n) phase-space we have to consider here. Hence we rely on appropriate magnetic coherent states $\psi_{R,n} \in L^2(\mathbb{R}^2)$ with Landau level index n, approximately localized around a guiding center position R. We associate to γ a Husimi function

$$m_\gamma(R, n) \propto \left\langle \psi_{R,n}, \gamma \psi_{R,n} \right\rangle$$

and study the dynamics thereof. A related approach, based on (x, p) coherent states, has been implemented previously in [12, 13, 14]. As it now stands, our method is rather demanding in terms of regularity of the potentials V and w (four bounded derivatives essentially). This can be improved if the boundedness of some moments of the magnetic kinetic energy are propagated in time (see Remark 5.2 below). We cannot prove this at present: we are in effect dealing with long-time asymptotics, for which it is difficult to keep moments under control, even at the level of the classical Vlasov equation.

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2. Main results

2.1. **Model and scaling.**

It will be convenient to work in a slightly different scaling than sketched in the introduction. We set this up first.

Notation 2.1 (Model).
We work on $\mathbb{R}^2$. The one body kinetic energy operator is the magnetic Laplacian

$$\mathscr{L}_b := (i\hbar\nabla + bA)^2$$

With
$$\mathrm{Dom}\left(\mathscr{L}_b\right) := \left\{\psi \in L^2\left(\mathbb{R}^2\right) | \mathscr{L}_b \psi \in L^2\left(\mathbb{R}^2\right)\right\}$$
We work in symmetric gauge, namely the vector potential is
$$A = \frac{1}{2} X^{\perp} \tag{2.1}$$
where X is the position multiplication operator in $\mathbb{R}^2$. We denote by b the magnetic field amplitude, associated to the magnetic length
$$l_b := \sqrt{\frac{\hbar}{b}}$$
Let V be the external potential and w the interaction potential, assumed to be radial.

We study a solution $\gamma \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ to the Hartree equation
$$\boxed{i\hbar\partial_t \gamma = \left[\mathscr{L}_b + V + w \star \rho_\gamma, \gamma\right]} \tag{2.2}$$
where $\mathcal{L}^p$ is the p-th Schatten class,
$$\rho_\gamma(t,x) := \gamma(t)(x,x) \tag{2.3}$$
the density associated to γ that we identify with its integral kernel. We will denote
$$H_b(t) := \mathscr{L}_b + V + \frac{1}{2} w \star \rho_{\gamma_b(t)}. \tag{2.4}$$
◇

Our goal is to obtain from the Hartree equation (2.2) the following drift equation for a density $\rho : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$,
$$\partial_t \rho + \nabla^{\perp}(V + w \star \rho)) \cdot \nabla \rho = 0. \tag{2.5}$$
We will denote
$$\mathrm{DRIFT}_\rho(\mu)(t,z) := \partial_t \mu(t,z) + \nabla^{\perp}\left(V + w \star \rho(t)\right)(z) \cdot \nabla \mu(t,z)$$
so that our target equation (2.5) takes the form $\mathrm{DRIFT}_\rho(\rho) = 0$.

Our plan is to examine a truly large magnetic field regime where all the terms in the Hamiltonian (2.4) are of order 1. As recalled in Section 3 below, the order of magnitude of the kinetic energy is $\hbar b$, which we will henceforth fix to unity. As discussed in the introduction, the time-scale we work on is of order b. Since we consider fermionic particles, constraints on the density matrix are imposed to enforce the Pauli exclusion principle. We summarize these conventions below:

Notation 2.2 (Scaling).
We work in a large magnetic field/semi-classical limit
$$b \to +\infty, \quad \hbar \underset{b\to\infty}{\to} 0$$
such that the magnetic kinetic energy is of order 1:
$$\hbar b \underset{b\to\infty}{\to} 1. \tag{2.6}$$
Let $\gamma \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2(\mathbb{R}^2)\right)\right)$, such that
$$\mathrm{Tr}\left[\gamma(0)\right] = 1 \quad \text{and} \quad 0 \le \gamma(0) \le 2\pi l_b^2 = 2\pi\frac{\hbar}{b} \tag{2.7}$$
define the time rescaled density matrix
$$\forall t \in \mathbb{R}_+, \quad \gamma_b(t) := \gamma(bt) \tag{2.8}$$

⋄

If γ satisfies (2.2), the equation for the time-rescaled density matrix is

$$\partial_t \gamma_b = \frac{b}{i\hbar}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b\right] = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b\right] \tag{2.9}$$

The Pauli principle $\gamma_b \leq 2\pi l_b^2$ (which propagates in time, see Lemma 3.8 below) guarantees that the system occupies a volume of order 1 in the limit

$$l_b \underset{b\to\infty}{\to} 0$$

Indeed it is known, [30, Chapter 3] or [40, Subsection I.4], that the degeneracy per area inside a Landau level is of order l_b^{-2}. A typical fermionic state satisfying (2.7) is a projection onto a N-body Slater determinant of N orthonormal one body wave-functions with

$$N := \mathcal{O}\left(\frac{1}{2\pi l_b^2}\right)$$

Such a N-particles state occupies a volume of order

$$\frac{N}{l_b^{-2}} = \mathcal{O}(1)$$

Hence with (2.6) this confirms that all the terms in the Hamiltonian $\mathscr{L}_b + V + w\star\rho_\gamma$ are of order 1. As a remark, we give an equivalent formulation of this scaling, making connections to the introductory section. If one takes exactly $\hbar = 1/b$, then (2.9) is equivalent to

$$i\partial_t \gamma = \left[\left(i\nabla + b^2 A\right) + b^2(V + w \star \rho_{\gamma_b}), \gamma_b\right]$$

In other words with the new scaling

$$\widetilde{b} := b^2$$
$$\widetilde{\gamma_b} := \frac{b^2}{2\pi}\gamma$$

we have

$$\mathrm{Tr}\left[\widetilde{\gamma_b}\right] = \frac{\widetilde{b}}{2\pi}, \quad \widetilde{\gamma_b} \leq 1$$
$$i\partial_t \widetilde{\gamma_b} = \left[\left(i\nabla + \widetilde{b}A\right)^2 + \widetilde{b}V + w \star \rho_{\widetilde{\gamma_b}}, \widetilde{\gamma_b}\right]$$

where all the terms in the Hamiltonian $\left(i\nabla + \widetilde{b}A\right)^2 + \widetilde{b}V + w \star \rho_{\widetilde{\gamma_b}}$ are of order $\widetilde{b}$. The above is the equivalent of the scaling "particle number proportional to magnetic field", $\widetilde{b} \propto N$ originally studied in [37] at the level of ground states, where a magnetic Thomas-Fermi theory emerges as the relevant effective description.

2.2. **Results.** We may now state our main results. We first prove that the density of the Hartree solution almost satisfies the weak form of the drift equation:

Theorem 2.3 (Dynamics of the Hartree solution).
Let $V, w \in W^{4,\infty}\left(\mathbb{R}^2\right)$ and $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ be a solution of (2.9) *with initial datum satisfying*

$$0 \le \gamma_b(0) \le 2\pi l_b^2, \quad \operatorname{Tr}\left[\gamma_b(0)\right] = 1, \quad \operatorname{Tr}\left[\gamma_b(0) H_b(0)\right] < C$$

with C independent of b. Let the associated drift operator be as in (2.5)*. Then, $\forall \varphi \in C^\infty\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$ with compact support*

$$\left| \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}(t,z) \mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t,z) dt dz - \int_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}(0,z) dz \right| \le C(\varphi, V, w) l_b^{\frac{2}{7}}$$

for some fixed constant $C(\varphi, V, w)$.

The proof of the above consists of two main parts:

• We define a semi-classical measure on phase-space approximating the exact quantum density. Using the vortex coherent state $\psi_{z,n}$ (approximately) localized around the position $z \in \mathbb{C} \leftrightarrow \mathbb{R}^2$ and (exactly) localized in the n-th Landau level of the magnetic kinetic energy (see Section 3.2 below for more details) we form the Husimi function

$$(z,n) \mapsto \frac{1}{2\pi l_b^2} \left\langle \psi_{z,n}, \gamma_b \psi_{z,n} \right\rangle =: m_{\gamma_b}(z,n).$$

Selecting a suitably large cut-off $M \gg 1$ for the Landau level index we mimic the true density by summing the above for $n \le M$

$$\rho_{\gamma_b}^{sc,\le M} := \sum_{n \le M} m_{\gamma_b}(z,n) \simeq \rho_{\gamma_b} \tag{2.10}$$

under suitable, mild assumptions.

• Combining the Hartree equation (2.2) with the algebraic properties of vortex coherent states we find that $\rho_{\gamma_b}^{sc,\le M}$ approximately solves (for suitably large b and tuned $M \gg 1$) the drift equation (2.5). The control of the implied error terms is the core analytical part of the proof.

Next, using appropriate stability estimates for solutions of the limiting drift equation, we can lift the above theorem to an estimate between ρ_{γ_b} and the classical solution. We denote W_1 the Monge-Kantorovich-Wasserstein (MKW) metric

$$W_1(\mu,\nu) := \inf_{\pi \in \Gamma(\mu,\nu)} \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x-y| \, d\pi(x,y) = \sup_{\|\nabla\varphi\|_{L^\infty\left(\mathbb{R}^2\right)} \le 1} \left| \int_{\mathbb{R}^2} \varphi d\left(\mu - \nu\right) \right|$$

where $\Gamma(\mu,\nu)$ is the set of couplings between $\mu, \nu \in \mathcal{P}\left(\mathbb{R}^2\right)$, namely $\mathcal{P}\left(\mathbb{R}^4\right) \ni \pi \in \Gamma(\mu,\nu)$ if

$$\int_{\mathbb{R}^2} \pi(x,y) dy = \mu(x), \quad \int_{\mathbb{R}^2} \pi(x,y) dx = \nu(y). \tag{2.11}$$

We then have

Theorem 2.4 (Convergence of densities).
We make the same assumptions as in Theorem 2.3, with in addition $\nabla w \in L^1\left(\mathbb{R}^2\right), w \in H^2\left(\mathbb{R}^2\right)$ and

$$\operatorname{Tr}\left[\gamma_b(0) |X|^p\right] < C \tag{2.12}$$

independently of b, for some $p > 7$.

Let $\rho \in L^\infty\left(\mathbb{R}_+, L^1\left(\mathbb{R}^2\right)\right)$ solve the drift equation (2.5). *Then $\forall t \in \mathbb{R}_+$ and every test function φ over $\mathbb{R}^2$*

$$\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b}(t) - \rho(t)\right)\right| \leq \widetilde{C}(p,t,V,w)\left(\|\varphi\|_{W^{1,\infty}} + \|\nabla\varphi\|_{L^2}\right)\left(W_1\left(\rho_{\gamma_b}(0), \rho(0)\right) + l_b^{\min\left(2\frac{p-7}{4p-7}, \frac{2}{7}\right)}\right) \tag{2.13}$$

with

$$\widetilde{C}(p,t,V,w) := \left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}$$
$$+ e^{2\left(\|w\|_{W^{2,\infty}} + \|V\|_{W^{2,\infty}}\right)t}\Bigg(1 + C(p)\left(1 + \mathrm{Tr}\left[\gamma_b(0)\,|X|^p\right]\right)\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)$$
$$+ Ct^2\left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}}e^{\left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right)t}\right)\left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}} + \|w\|_{H^2}\right)$$
$$\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\Bigg)$$

Theorem 2.4 follows from a stability estimate à la Dobrushin [17] for classical Hamiltonian equations. We treat the error between the quantum and classical equations obtained from Theorem 2.3 as a source term for the limiting equation, and show that the stability theory of the latter (as reviewed e.g. in [23, Section 1.4]) survives this addition. When considering the limit of the semi-classical density (2.10), an estimate is obtained directly in Wasserstein-1 metric, see Proposition 6.4 below. The additional initial trapping assumption (2.12), and the slight modification of the norm in which the convergence is measured (2.13), arise when vindicating the approximation in (2.10). As regards the additional assumption (2.12), we note that it is fairly natural for typical initial data. For example, equilibria of systems with an additional trapping external potential included in the Hamiltonian usually decay exponentially in space, in which case one may think that $p = \infty$ formally.

2.3. **Organisation of the paper.** Section 3 covers preliminaries, focusing on the magnetic Laplacian and the conserved properties of the dynamics. In Section 4, we introduce Husimi functions, the associated semi-classical densities and prove that they approximate the physical density. Then we study the dynamics of these semi-classical densities in Section 5 and prove they approximately follow the drift equation. The conclusion of the proof of Theorem 2.3 is given in Section 5.2 by combining this with the results of Section 4. We study perturbed classical flows associated with (2.5) in Section 6. This leads to the Dobrushin-like stability estimate allowing to conclude the proof of Theorem 2.4 in Section 6.1.

3. Landau quantization and the magnetic Hartree equation

We here recall the usual formalism for describing the magnetic Laplacian in terms of annihilation and creation operators. Further details about these operators and the properties of Landau levels are reviewed e.g. in [44] or [40] and references therein. This formalism provides a basis (3.1) of eigenstates indexed by two quantum numbers n and m, with n denoting the index describing the Landau level and m representing "angular momentum minus Landau level index". To obtain a projector on a point in phase space, we use coherent states for a fixed n. In two dimensions, the complex parameter in the definition of coherent states can be identified with a position. Consequently, following e.g. [9, 11, 10], we construct a one-particle state localized at a specific point in space, see Definition 3.4. Then, we provide some properties of the associated projector. We conclude this section with a brief recap of the conserved quantities of the Hartree equation (2.2).

3.1. Landau quantization.

Notation 3.1 (Magnetic momentum and kinetic energy).
We denote by p_1, p_2 the coordinates of the magnetic momentum

$$\mathscr{P}_{\hbar,b} =: \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} =: -\begin{pmatrix} i\hbar\partial_1 + bA_1 \\ i\hbar\partial_2 + bA_2 \end{pmatrix}$$

with $A = \frac{X^\perp}{2}$. Define then the annihilation, creation and number operators respectively as

$$a := \frac{p_1 + ip_2}{\sqrt{2\hbar b}}, \quad a^\dagger := \frac{p_1 - ip_2}{\sqrt{2\hbar b}}, \quad \mathcal{N} := a^\dagger a$$

◇

We have the commutation relations:

$$\begin{aligned} \left[p_1, p_2\right] &= i\hbar b \\ \left[a, a^\dagger\right] &= \mathbb{1} \text{ (canonical commutation relation)} \end{aligned}$$

and may express the magnetic Laplacian as

$$\mathscr{L}_b = 2\hbar b \left(\mathcal{N} + \frac{\mathrm{Id}}{2} \right)$$

Notation 3.2 (Landau levels).
We define the n^{th} Landau level as the eigenspace associated to $n \in \mathbb{N}$:

$$\mathrm{nLL} := \left\{ \psi \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \text{ such that } \mathcal{N}\psi = n\psi \right\}$$

The ground level, denoted LLL for *Lowest Landau Level* has energy $E_0 = \hbar b$. ◇

The Landau levels are isomorphic, and the operator $a^\dagger/\sqrt{n+1}$ is a unitary mapping from nLL to (n+1)LL of inverse $a/\sqrt{n+1}$. Therefore we may, using $a^\dagger$, extend a basis of LLL to higher Landau levels. The Lowest Landau level consists of holomorphic functions pondered by a Gaussian factor, see e.g. [44].

The Landau level quantization of the kinetic energy corresponds to the quantization of the cyclotron orbit. To complete this aspect, we associate an operator to the motion of the guiding center of the orbit.

Notation 3.3 (Guiding center oscillator).
For the rest of the text, we will identify a vector

$$x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

with the complex notation, in bold, $\mathbf{x} := x_1 + ix_2$. We introduce the following position operators

$$\begin{aligned} r &:= \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} := \frac{\mathscr{P}^\perp_{\hbar,b}}{b} = \frac{1}{b}\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} \\ R &:= X - r \end{aligned}$$

and associate to R the creation and annihilation operators

$$c = \frac{R_1 - iR_2}{\sqrt{2} l_b} \quad c^\dagger = \frac{R_1 + iR_2}{\sqrt{2} l_b}$$

◇

The operator r represents the position of a particle in the center of orbit frame. The classical physics meaning of this definition is that, during cyclotron motion, the momentum is perpendicular to the position relative to the center to the orbit. Electrons are describing clockwise orbits, thus the momentum rotated of $\pi/2$ gives us r. Moreover, r is related to the quantization of the cyclotron pulsation of the orbit because

$$a = \frac{p_1 + ip_2}{\sqrt{2\hbar b}} = \frac{r_2 - ir_1}{\sqrt{2}l_b} = \frac{-i\mathbf{r}}{\sqrt{2}l_b}, \quad a^\dagger = \frac{i\overline{\mathbf{r}}}{\sqrt{2}l_b}.$$

From the definition of r, the position R of the orbit center is indeed

$$X = R + r$$

and related to the second harmonic oscillator

$$c = \frac{\overline{\mathbf{R}}}{\sqrt{2}l_b}, \quad c^\dagger = \frac{\mathbf{R}}{\sqrt{2}l_b}.$$

The components of r, R and X commute with one another. Moreover

$$\begin{aligned}
&\left[r_1, r_2\right] = il_b^2 \\
&\left[R_1, R_2\right] = -il_b^2 \\
&\left[c, c^\dagger\right] = \mathrm{Id} \\
&[a, c] = \left[a, c^\dagger\right] = \left[a^\dagger, c\right] = \left[a^\dagger, c^\dagger\right] = 0
\end{aligned}$$

We therefore have two independent harmonic oscillators. By successively applying the creation operators $a^\dagger$ and $b^\dagger$ we obtain the desired eigenbasis of the magnetic Laplacian. In symmetric gauge (2.1), the family defined by

$$\varphi_{n,m} := \frac{\left(a^\dagger\right)^n \left(c^\dagger\right)^m}{\sqrt{n!m!}} \varphi_{0,0} \tag{3.1}$$

with

$$\varphi_{0,0} = \frac{1}{\sqrt{2\pi}l_b} e^{\frac{-|z|^2}{4l_b^2}}$$

is an orthonormal Hilbert basis of $L^2\left(\mathbb{R}^2\right)$. The full expression, see [9, 11, 10, 44], is

$$\varphi_{n,m}(x) = \frac{\left(\left(-2il_b^2\partial_{\mathbf{x}} + i\overline{\mathbf{x}}\right)^n \mathbf{x}^m\right)}{\sqrt{\pi n!m!}\left(\sqrt{2}l_b\right)^{n+m+1}} e^{\frac{-|x|^2}{4l_b^2}} \tag{3.2}$$

The orthogonal projector on the n-th Landau level from Notation 3.2 is recovered as

$$\Pi_n := \sum_{m\in\mathbb{N}} \left|\varphi_{n,m}\right\rangle \left\langle\varphi_{n,m}\right|.$$

3.2. **Coherent states.** We next define coherent states in order to have wave functions localized at a precise point in the phase-space "position of the orbit center × Landau level index ".

Definition 3.4 (Vortex coherent states).
Let $z \in \mathbb{C} \leftrightarrow \mathbb{R}^2$, $n \in \mathbb{N}$. We define the associated coherent state

$$\psi_{z,n} := e^{\frac{\overline{z}b^\dagger - zb}{\sqrt{2}l_b}} \varphi_{n,0}$$

and the associated projector

$$\Pi_{z,n} := \left|\psi_{z,n}\right\rangle \left\langle\psi_{z,n}\right|. \tag{3.3}$$

We let the localised projector

$$\Pi_z = \sum_{n\in\mathbb{N}} \Pi_{z,n}$$

Finally we define for $M \in \mathbb{N}$ or $N_1 \leq N_2$ the truncated projectors

$$\Pi_{N_1:N_2} := \sum_{n=N_1}^{N_2} \Pi_n, \quad \Pi_{\leq M} := \Pi_{0:M}, \quad \Pi_{z,\leq M} := \sum_{n=0}^{N} \Pi_{z,n}$$

with similar definitions for $\Pi_{>M}$ and $\Pi_{>M,z}$. ⋄

We will rely heavily on the following closure relations [9, 44]:

Lemma 3.5 (**Coherent states partition of unity).**
With Definition 3.4,

$$\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2} \Pi_{z,n} dz = \Pi_n \tag{3.4}$$

$$\frac{1}{2\pi l_b^2}\sum_{n\in\mathbb{N}}\int_{\mathbb{R}^2} \Pi_{z,n} dz = \mathbb{1}_{L^2(\mathbb{R}^2)} \tag{3.5}$$

Moreover, for any $n_1, n_2, p_1, p_2 \in \mathbb{N}$

$$\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2} z^{p_1}\left|\psi_{z,n_1}\right\rangle\left\langle\psi_{z,n_2}\right|\overline{z}^{p_2} dz = \left(\sqrt{2}l_b\right)^{p_1+p_2}\sum_{m=\max(p_1,p_2)}^{\infty}\frac{m!\left|\varphi_{n_1,m-p_1}\right\rangle\left\langle\varphi_{n_2,m-p_2}\right|}{\sqrt{(m-p_1)!\,(m-p_2)!}}. \tag{3.6}$$

Proof. The first two points are standard closure identities for coherent states, whose proofs can be found in [9, 44]. For the last point, with the change of variable $m := m_1+p_1 = m_2+p_2$, we compute

$$\begin{aligned}
&\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2} z^{p_1}\left|\psi_{z,n_1}\right\rangle\left\langle\psi_{z,n_2}\right|\overline{z}^{p_2} dz \\
=&\frac{\left(\sqrt{2}l_b\right)^{p_1+p_2}}{2\pi l_b^2}\sum_{m_1,m_2\in\mathbb{N}}\frac{1}{\sqrt{m_1!m_2!}}\int_{\mathbb{R}^2}\left(\frac{z}{\sqrt{2}l_b}\right)^{m_1+p_1}\left(\frac{\overline{z}}{\sqrt{2}l_b}\right)^{m_2+p_2} e^{-\frac{|z|^2}{2l_b^2}} dz\left|\varphi_{n_1,m_1}\right\rangle\left\langle\varphi_{n_2,m_2}\right| \\
=&\frac{\left(\sqrt{2}l_b\right)^{p_1+p_2}}{2\pi l_b^2}\sum_{m=\max(p_1,p_2)}^{\infty}\frac{1}{\sqrt{(m-p_1)!\,(m-p_2)!}}\int_{\mathbb{R}^2}\left(\frac{|z|^2}{2l_b^2}\right)^{m} e^{-\frac{|z|^2}{2l_b^2}} dz\left|\varphi_{n_1,m-p_1}\right\rangle\left\langle\varphi_{n_2,m-p_2}\right| \\
=&\left(\sqrt{2}l_b\right)^{p_1+p_2}\sum_{m=\max(p_1,p_2)}^{\infty}\frac{m!\left|\varphi_{n_1,m-p_1}\right\rangle\left\langle\varphi_{n_2,m-p_2}\right|}{\sqrt{(m-p_1)!\,(m-p_2)!}}
\end{aligned}$$

□

Note that $\psi_{z,n}$ is localised around z in the sense that

$$\overline{\mathbf{R}}\psi_{z,n} = \overline{\mathbf{z}}\psi_{z,n}$$

with fluctuations of order l_b (as can be seen from the next lemma). The following explicit expressions will be important for us:

Lemma 3.6 (Expression of vortex coherent states).
We have

$$\psi_{z,n}(x) = \frac{i^n}{\sqrt{2\pi n!}l_b}\left(\frac{\mathbf{x} - \mathbf{z}}{\sqrt{2}l_b}\right)^n e^{-\frac{|x-z|^2-2iz^\perp\cdot x}{4l_b^2}} \tag{3.7}$$

$$\Pi_{z,n}(x,y) = \frac{1}{2\pi n! l_b^2}\left(\frac{\left(\overline{\mathbf{x} - \mathbf{z}}\right)(\mathbf{y} - \mathbf{z})}{2l_b^2}\right)^n e^{-\frac{|x-z|^2+|y-z|^2-2iz^\perp\cdot(x-y)}{4l_b^2}}$$

$$\Pi_z(x,y) = \frac{1}{2\pi l_b^2} e^{-\frac{|x-y|^2-2i\left(x^\perp\cdot y+2z^\perp\cdot(x-y)\right)}{4l_b^2}}. \tag{3.8}$$

Consequently

$$\nabla_z^\perp \Pi_z(x,y) = \frac{y-x}{il_b^2}\Pi_z(x,y) \tag{3.9}$$

or, as an operator identity,

$$\nabla_z^\perp \Pi_z = \frac{1}{il_b^2}\left[\Pi_z, X\right] \tag{3.10}$$

We refer to [9] again for the derivation of the above exact expressions, from which (3.9) immediately follows. The latter formula will play a key role in the computation of the spacial derivative of the density in Section 5 below. We will also rely heavily on an approximation thereof applying to the truncated projector.

Lemma 3.7 (Spatial derivatives of coherent state projectors).
For the localized projector (3.3) *we have*

$$\nabla_z^\perp \Pi_{z,n} = \frac{1}{il_b^2}\left[\Pi_{z,n}, X\right] - \frac{\sqrt{n+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{z,n}\right\rangle\left\langle\psi_{z,n+1}\right|\\ \left|\psi_{z,n+1}\right\rangle\left\langle\psi_{z,n}\right|\end{pmatrix}$$
$$+\frac{\sqrt{n}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{z,n-1}\right\rangle\left\langle\psi_{z,n}\right|\\ \left|\psi_{z,n}\right\rangle\left\langle\psi_{z,n-1}\right|\end{pmatrix} \tag{3.11}$$

and, summing over $n \leq M$,

$$\nabla_z^\perp \Pi_{z,\leq M} = \frac{1}{il_b^2}\left[\Pi_{z,\leq M}, X\right] - \frac{\sqrt{M+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{z,M}\right\rangle\left\langle\psi_{z,M+1}\right|\\ \left|\psi_{z,M+1}\right\rangle\left\langle\psi_{z,M}\right|\end{pmatrix}. \tag{3.12}$$

Proof. Starting from (3.7) we have

$$\psi_{z,n}(x) = \frac{-i}{\sqrt{n}}\frac{\overline{\mathbf{z} - \mathbf{x}}}{\sqrt{2}l_b}\psi_{n-1,z}(x), \quad \overline{\psi_{z,n}(y)} = \frac{i}{\sqrt{n}}\frac{\mathbf{z} - \mathbf{y}}{\sqrt{2}l_b}\overline{\psi_{n-1,z}(y)}.$$

Hence

$$\begin{aligned}\Pi_{z,n}(x,y) =& \psi_{z,n}(x)\overline{\psi_{z,n}(y)} = \frac{1}{2\pi n! l_b^2}\left(\frac{\left(\overline{\mathbf{x}-\mathbf{z}}\right)(\mathbf{y}-\mathbf{z})}{2l_b^2}\right)^n e^{-\frac{|x-z|^2+|y-z|^2-2iz^\perp\cdot(x-y)}{4l_b^2}}\\ =& \frac{1}{2\pi n! l_b^2}\left(\frac{\left(\overline{\mathbf{x}-\mathbf{z}}\right)(\mathbf{y}-\mathbf{z})}{2l_b^2}\right)^n e^{-\frac{2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})}{4l_b^2}}\end{aligned}$$

and we deduce

$$\begin{aligned}\partial_{\mathbf{z}}\Pi_{z,n}(x,y) =& \frac{\overline{\mathbf{y}-\mathbf{z}}}{2l_b^2}\Pi_{z,n}(x,y) + \frac{\overline{\mathbf{z}-\mathbf{x}}}{2l_b^2}\Pi_{n-1,z}(x,y)\\ =& \frac{\overline{\mathbf{y}-\mathbf{x}}}{2l_b^2}\psi_{z,n}(x)\overline{\psi_{z,n}(y)} + \frac{\overline{\mathbf{x}-\mathbf{z}}}{2l_b^2}\psi_{z,n}(x)\overline{\psi_{z,n}(y)} + \frac{\overline{\mathbf{z}-\mathbf{x}}}{2l_b^2}\psi_{n-1,z}(x)\overline{\psi_{n-1,z}(y)}\\ =& \frac{\overline{\mathbf{y}-\mathbf{x}}}{2l_b^2}\psi_{z,n}(x)\overline{\psi_{z,n}(y)} - i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\psi_{n+1,z}(x)\overline{\psi_{z,n}(y)} + i\frac{\sqrt{n}}{\sqrt{2}l_b}\psi_{z,n}(x)\overline{\psi_{n-1,z}(y)}\end{aligned}$$

together with

$$\begin{aligned}\partial_{\overline{\mathbf{z}}}\Pi_{z,n}(x,y) =& \frac{\mathbf{x}-\mathbf{z}}{2l_b^2}\Pi_{z,n}(x,y) + \frac{\mathbf{z}-\mathbf{y}}{2l_b^2}\Pi_{n-1,z}(x,y)\\ =& \frac{\mathbf{x}-\mathbf{y}}{2l_b^2}\psi_{z,n}(x)\overline{\psi_{z,n}(y)} + \frac{\mathbf{y}-\mathbf{z}}{2l_b^2}\psi_{z,n}(x)\overline{\psi_{z,n}(y)} + \frac{\mathbf{z}-\mathbf{y}}{2l_b^2}\psi_{n-1,z}(x)\overline{\psi_{n-1,z}(y)}\\ =& \frac{\mathbf{x}-\mathbf{y}}{2l_b^2}\psi_{z,n}(x)\overline{\psi_{z,n}(y)} + i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\psi_{z,n}(x)\overline{\psi_{n+1,z}(y)} - i\frac{\sqrt{n}}{\sqrt{2}l_b}\psi_{n-1,z}(x)\overline{\psi_{z,n}(y)}.\end{aligned}$$

This leads to

$$\begin{aligned}\partial_{z_1}\Pi_{z,n}(x,y) =& \left(\partial_{\mathbf{z}} + \partial_{\overline{\mathbf{z}}}\right)\Pi_{z,n}(x,y)\\ =& i\frac{x_2-y_2}{l_b^2}\Pi_{z,n}(x,y) + i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\left(\psi_{z,n}(x)\overline{\psi_{n+1,z}(y)} - \psi_{n+1,z}(x)\overline{\psi_{z,n}(y)}\right)\\ &- i\frac{\sqrt{n}}{\sqrt{2}l_b}\left(\psi_{n-1,z}(x)\overline{\psi_{z,n}(y)} - \psi_{z,n}(x)\overline{\psi_{n-1,z}(y)}\right)\end{aligned}$$

and

$$\begin{aligned}\partial_{z_2}\Pi_{z,n}(x,y) =& i\left(\partial_{\mathbf{z}} - \partial_{\overline{\mathbf{z}}}\right)\Pi_{z,n}(x,y)\\ =& i\frac{y_1-x_1}{l_b^2}\Pi_{z,n}(x,y) + \frac{\sqrt{n+1}}{\sqrt{2}l_b}\left(\psi_{z,n}(x)\overline{\psi_{n+1,z}(y)} + \psi_{n+1,z}(x)\overline{\psi_{z,n}(y)}\right)\\ &- \frac{\sqrt{n}}{\sqrt{2}l_b}\left(\psi_{n-1,z}(x)\overline{\psi_{z,n}(y)} + \psi_{z,n}(x)\overline{\psi_{n-1,z}(y)}\right)\end{aligned}$$

It follows that

$$\nabla_z^\perp \Pi_{z,n}(x,y) = i\frac{x-y}{l_b^2}\Pi_{z,n}(x,y) - \frac{\sqrt{n+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{z,n}(x)\overline{\psi_{z,n+1}(y)}\\ \psi_{z,n+1}(x)\overline{\psi_{z,n}(y)}\end{pmatrix}$$
$$+ \frac{\sqrt{n}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{z,n-1}(x)\overline{\psi_{z,n}(y)}\\ \psi_{z,n}(x)\overline{\psi_{z,n-1}(y)}\end{pmatrix},$$

which is (3.11). The summation over n cancels telescopic terms, leading to

$$\nabla_z^\perp \Pi_{z,\leq M}(x,y) = i\frac{x-y}{l_b^2}\Pi_{z,\leq M}(x,y) - \frac{\sqrt{M+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{z,M}(x)\overline{\psi_{z,M+1}(y)}\\ \psi_{z,M+1}(x)\overline{\psi_{z,M}(y)}\end{pmatrix}.$$

□

3.3. **Conservation properties.** We next state some basic properties of the Hartree dynamics (2.2).

Lemma 3.8 (**Conservation of mass and Pauli principle**).
Assume $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ *solves*

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w\star\rho_{\gamma_b(t)}, \gamma_b\right]$$

and satisfies

$$\mathrm{Tr}\left[\gamma_b(0)\right] = 1, \quad 0 \leq \gamma_b(0) \leq 2\pi l_b^2$$

then $\forall t \in \mathbb{R}_+$,

$$\mathrm{Tr}\left[\gamma_b(t)\right] = 1, \quad 0 \leq \gamma_b(t) \leq 2\pi l_b^2$$

Proof. This follows from the fact that the dynamics is Hamiltonian. □

We also have

Lemma 3.9 (**Energy conservation**).
Assume $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ *solves*

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w\star\rho_{\gamma_b(t)}, \gamma_b\right]$$

then

$$\frac{d}{dt}\mathrm{Tr}\left[\gamma_b(t)H_b(t)\right] = 0.$$

Moreover

$$\mathrm{Tr}\left[\gamma_b\mathscr{L}_b\right] \leq \left|\mathrm{Tr}\left[\gamma_b\left(\mathscr{L}_b + W\right)\right]\right| + \|W\|_{L^\infty}$$

Proof. The equation being Hamiltonian, the total energy is certainly conserved. Then, the kinetic energy is bounded by

$$\mathrm{Tr}\left[\gamma_b\mathscr{L}_b\right] = \mathrm{Tr}\left[\gamma_b\left(\mathscr{L}_b + W\right)\right] - \mathrm{Tr}\left[\gamma_b W\right] \leq \left|\mathrm{Tr}\left[\gamma_b\left(\mathscr{L}_b + W\right)\right]\right| + \|W\|_{L^\infty}$$

□

4. Husimi functions and semi-classical densities

In this section, we provide the first set of tools mentioned below Theorem 2.3, namely the construction and properties of Husimi functions and associated semi-classical densities.

4.1. **Semi-classical density.** Given a coherent state basis over a Hilbert space and a trace-class operator acting on the latter, the notion of Husimi function/lower symbol is fairly standard (see e.g. [43, Section 3.3] and references therein):

Definition 4.1 (Husimi function and associated density).
For $\gamma \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$, let

$$m_\gamma(z,n) := \frac{1}{2\pi l_b^2}\left\langle \psi_{z,n} \middle| \gamma \psi_{z,n}\right\rangle = \frac{1}{2\pi l_b^2}\mathrm{Tr}\left[\Pi_{z,n}\gamma\right] \tag{4.1}$$

with the associated semi-classical density

$$\rho^{sc}(z) := \sum_{n=0}^{\infty} m_\gamma(z,n).$$

For $M \subset \mathbb{N}$ such that $1 \ll M \ll l_b^{-2}$, we define the truncated version thereof

$$\rho_\gamma^{sc,\leq M}(z) := \frac{1}{2\pi l_b^2}\mathrm{Tr}\left[\gamma\Pi_{z,\leq M}\right] \tag{4.2}$$

◇

The parameter M represents the number of Landau levels we take into account for the semi-classical approximations. It will be important to have $M \gg 1$ when $b \to \infty$ to recover the true quantum density ρ_γ of a general γ (with reasonable magnetic kinetic energy). On the other hand, the larger n the less the coherent state $\psi_{z,n}$ is localized around z, making the approximation less efficient. For this reason we will mostly use the truncated (4.2) for a suitable $1 \ll M \ll l_b^{-2}$, and use moments of the kinetic energy to discard the contribution to the density of Landau levels with index $n > M$.

The main estimate we will rely on is as follows:

Proposition 4.2 (Convergence of the truncated semi-classical density). ,
Let $k \geq 0, \gamma_b \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$ *and assume*

$$\mathrm{Tr}\left[\gamma_b\right] = 1, \quad 0 \leq \gamma_b \leq 2\pi l_b^2, \quad \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right] < \infty \tag{4.3}$$

then $\forall \varphi \in L^\infty\left(\mathbb{R}^2\right) \cap H^1\left(\mathbb{R}^2\right)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leq M}\right)\right| &\leq \|\varphi\|_{L^\infty} M^{-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\Pi_{>M}\mathscr{L}_b^k\right]} \\ &\quad + C\,\|\nabla\varphi\|_{L^2}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]} \cdot \begin{cases} M^{1-\frac{k}{2}} l_b & \text{if } k < 2 \\ \sqrt{\ln(M)}\, l_b & \text{if } k = 2 \\ l_b & \text{if } k > 2 \end{cases}\end{aligned} \tag{4.4}$$

The first term on the right-hand side of (4.4) corresponds to the contribution of high Landau levels. In our approach to the semi-classical limit we will rely solely on one moment of the magnetic kinetic energy being bounded uniformly in time, $k = 1$. Observe that then the error terms will be small if $\sqrt{M} l_b \ll 1$. This constraint has a physical meaning. Specifically, from the expression of the coherent state (3.7), we can infer that the characteristic localization length for particles in nLL is $\sqrt{n} l_b$. Therefore, $\rho_{\gamma_b}^{sc,\leq M}$ with $\sqrt{M} l_b \ll 1$ represents the semi-classical density of particles localized with precision better than what is aimed at in the classical equation. The fluctuations of the position operator will be small in the limit for this contribution.

We will have to test γ_b against multiplication operators by nice functions φ. Using the resolution of the identity from Lemma 3.5, the gist of the estimate consists in vindicating that

$$\frac{1}{2\pi l_b^2}\sum_{n\geq 0}\int_{\mathbb{R}^2}\iint_{\mathbb{R}^4}(\varphi(x)-\varphi(z))\,\gamma_b(x,y)\Pi_{z,n}(y,x)dxdydz \underset{b\to\infty}{\simeq} 0$$

using that the coherent projector's kernel

$$\Pi_{z,n}(x,y)=\psi_{z,n}(x)\overline{\psi_{z,n}(y)}$$

is strongly localized, for moderate values of n, around $z=y=x$. We start with a lemma that will deal with the part of the sum bearing on low Landau levels, A_z below playing the role of the multiplication operator by $\varphi(\bullet)-\varphi(z)$.

Lemma 4.3 (Bounds on expectations of truncated operators).
Let $\forall z\in\mathbb{R}^2, A_z$ be an operator on $L^2\left(\mathbb{R}^2\right)$, $k\leq 0, \gamma_b\in\mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$ and assume

$$\mathrm{Tr}\left[\gamma_b\right]=1, 0\leq\gamma_b\leq 2\pi l_b^2. \tag{4.5}$$

Then

$$\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\left|\mathrm{Tr}\left[A_z\gamma_b\Pi_{z,\leq M}\right]\right|dz\leq\frac{1}{2}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\left(\sum_{n=0}^{M}\frac{1}{(n+1)^k}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]dz\right)^{\frac{1}{2}} \tag{4.6}$$

and $\forall\varphi\in L^1\left(\mathbb{R}^2\right)\cap L^\infty\left(\mathbb{R}^2\right)$,

$$\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}|\varphi(z)|\left|\mathrm{Tr}\left[A_z\gamma_b\Pi_{z,\leq M}\right]\right|dz\leq\frac{\sqrt{\|\varphi\|_{L^1}\|\varphi\|_{L^\infty}}}{2}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\cdot\left(\sum_{n=0}^{M}\frac{\left\|\mathrm{Tr}\left[\left|A_\bullet\right|^2\Pi_{\bullet,n}\right]\right\|_{L^\infty}}{(n+1)^k}\right)^{\frac{1}{2}} \tag{4.7}$$

Proof. The main idea is to exploit the sum over Landau levels to introduce moments of the kinetic energy using

$$\begin{aligned}\sum_{n=0}^{M}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]&\leq\sum_{n\in\mathbb{N}}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]=\sum_{n\in\mathbb{N}}\left(\frac{n+1}{2\hbar b\left(n+\frac{1}{2}\right)}\right)^k\mathrm{Tr}\left[\gamma_b\Pi_n\mathscr{L}_b^k\right]\\&\leq\sum_{n\in\mathbb{N}}\mathrm{Tr}\left[\gamma_b\Pi_n\mathscr{L}_b^k\right]=\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\end{aligned} \tag{4.8}$$

Applying the Cauchy-Schwarz inequality, using $\Pi_{z,n}^2 = \Pi_{z,n}$ and (4.5) followed by Young's inequality we find

$$\begin{aligned}\left|\mathrm{Tr}\left[A_z\gamma_b\Pi_{z,\le M}\right]\right| &= \left|\sum_{n=0}^{M}\mathrm{Tr}\left[A_z\gamma_b\Pi_{z,n}\right]\right| \le \sum_{n=0}^{M}\sqrt{\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]}\sqrt{\mathrm{Tr}\left[\gamma_b^2\Pi_{z,n}\right]}\\ &\le\sqrt{2\pi l_b^2}\sum_{n=0}^{M}\sqrt{\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]}\sqrt{\mathrm{Tr}\left[\gamma_b\Pi_{z,n}\right]}\\ &\le\sqrt{\frac{\pi}{2}}l_b\sum_{n=0}^{M}\left(\frac{1}{\epsilon_n}\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]+\epsilon_n\mathrm{Tr}\left[\gamma_b\Pi_{z,n}\right]\right)\end{aligned} \tag{4.9}$$

where we will choose $\epsilon_n := \epsilon(n+1)^k$ for some $\epsilon > 0$. Integrating in z, using (3.4) and inserting (4.8) gives

$$\begin{aligned}&\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\left|\mathrm{Tr}\left[A_z\gamma_b\Pi_{z,\le M}\right]\right|dz\\ \le&\sqrt{\frac{\pi}{2}}l_b\sum_{n=0}^{M}\left(\frac{1}{2\pi l_b^2\epsilon_n}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]dz+\epsilon_n\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right)\\ \le&\sqrt{\frac{\pi}{2}}l_b\left(\sum_{n=0}^{M}\frac{1}{2\pi l_b^2\epsilon(n+1)^k}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]dz+\epsilon\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)\end{aligned}$$

Choosing now

$$\epsilon := \left(\frac{1}{2\pi l_b^2}\sum_{n=0}^{M}\frac{1}{(n+1)^k}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]dz\frac{1}{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\right)^{\frac{1}{2}}$$

leads to (4.6):

$$\begin{aligned}&\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\left|\mathrm{Tr}\left[A_z\gamma_b\Pi_{z,\le M}\right]\right|dz\\ \le&\sqrt{\frac{\pi}{2}}l_b\left(\frac{1}{2\pi l_b^2}\sum_{n=0}^{M}\frac{1}{(n+1)^k}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]dz\right)^{\frac{1}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\\ =&\frac{1}{2}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\left(\sum_{n=0}^{M}\frac{1}{(n+1)^k}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]dz\right)^{\frac{1}{2}}\end{aligned}$$

Starting again from (4.9),

$$\begin{aligned}&\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}|\varphi(z)|\left|\mathrm{Tr}\left[A_z\gamma_b\Pi_{z,\leq M}\right]\right|dz\\ \leq&\sqrt{\frac{\pi}{2}}l_b\left(\sum_{n=0}^{M}\frac{1}{2\pi l_b^2\epsilon(n+1)^k}\int_{\mathbb{R}^2}|\varphi(z)|\,\mathrm{Tr}\left[\left|A_z\right|^2\Pi_{z,n}\right]dz+\epsilon\,\|\varphi\|_{L^\infty}\,\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)\\ \leq&\sqrt{\frac{\pi}{2}}l_b\left(\sum_{n=0}^{M}\frac{\|\varphi\|_{L^1}\left\|\mathrm{Tr}\left[\left|A_\bullet\right|^2\Pi_{\bullet,n}\right]\right\|_{L^\infty}}{2\pi l_b^2\epsilon(n+1)^k}+\epsilon\,\|\varphi\|_{L^\infty}\,\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right),\end{aligned}$$

we obtain (4.7) by choosing instead

$$\epsilon:=\sqrt{\frac{\|\varphi\|_{L^1}}{2\pi l_b^2\,\|\varphi\|_{L^\infty}}\sum_{n=0}^{M}\frac{\left\|\mathrm{Tr}\left[\left|A_\bullet\right|^2\Pi_{\bullet,n}\right]\right\|_{L^\infty}}{(n+1)^k}\frac{1}{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}}$$

□

We may next proceed to the

Proof of Proposition 4.2. For $\varphi\in C_c^\infty\left(\mathbb{R}^2\right)$, we write

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi\left(\rho_{\gamma_b}-\rho_{\gamma_b}^{sc,\leq M}\right)=&\mathrm{Tr}\left[\varphi\gamma_b\right]-\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\varphi(z)\mathrm{Tr}\left[\Pi_{z,\leq M}\gamma_b\right]dz\\ =&\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[(\varphi-\varphi(z))\,\Pi_{z,\leq M}\gamma_b\right]dz+\mathrm{Tr}\left[\varphi\Pi_{>M}\gamma_b\right]\end{aligned}\tag{4.10}$$

Step 1, low Landau levels. We will apply Lemma 4.3 to $A_z:=\varphi-\varphi(z)$. Using the change of variables

$$\frac{x-z}{\sqrt{2}l_b}\mapsto x$$

and Taylor's theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{z,n}\right]dz =& \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} |\varphi(x)-\varphi(z)|^2\,\Pi_{z,n}(x,x)dxdz \\
=&\frac{1}{2\pi n! l_b^2}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} |\varphi(x)-\varphi(z)|^2 \left|\frac{x-z}{\sqrt{2}l_b}\right|^{2n} e^{-\frac{|x-z|^2}{2l_b^2}}dxdz \\
=&\frac{1}{\pi n!}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\left|\varphi\left(z+\sqrt{2}l_b x\right)-\varphi(z)\right|^2 |x|^{2n} e^{-|x|^2}dxdz \\
=&\frac{1}{\pi n!}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\left|\int_0^1 \nabla\varphi(z+\sqrt{2}l_b xs)\cdot\sqrt{2}l_b x ds\right|^2 |x|^{2n} e^{-|x|^2}dxdz \\
\le&\frac{2l_b^2}{\pi n!}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\int_0^1\left|\nabla\varphi(z+\sqrt{2}l_b xs)\right|^2 |x|^{2(n+1)} e^{-|x|^2}dsdxdz \\
\le&\frac{2l_b^2}{\pi n!}\|\nabla\varphi\|_{L^2}^2\int_{\mathbb{R}^2}|x|^{2(n+1)}e^{-|x|^2}dx = 2\|\nabla\varphi\|_{L^2}^2 (n+1)l_b^2
\end{aligned} \tag{4.11}$$

Introducing the notation

$$p_\lambda(x) := x^{-\lambda}\mathbb{1}_{\lambda<0} + \ln(x)\mathbb{1}_{\lambda=0} + \mathbb{1}_{\lambda>0} \tag{4.12}$$

we have the asymptotics

$$\sum_{n=0}^{M}\frac{1}{(n+1)^{k-1}} = \begin{cases} \mathcal{O}\left(M^{2-k}\right) & \text{if } k<2 \\ \mathcal{O}(\ln(M)) & \text{if } k=2 \\ \mathcal{O}(1) & \text{if } k>2 \end{cases} = \mathcal{O}\left(p_{k-2}(M)\right)$$

Hence (4.3) gives

$$\sum_{n=0}^{M}\frac{1}{(n+1)^k}\int_{\mathbb{R}^2}\mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{z,n}\right]dz \le 2l_b^2\|\nabla\varphi\|_{L^2}^2\sum_{n=0}^{M}(n+1)^{1-k} = Cl_b^2\|\nabla\varphi\|_{L^2}^2 p_{k-2}(M)$$

Using (4.6) we obtain

$$\begin{aligned}
&\left|\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[(\varphi-\varphi(z))\,\gamma_b\Pi_{z,\le M}\right]dz\right| \\
\le&\frac{1}{2}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\left(\frac{1}{2\pi l_b^2}\sum_{n=0}^{M}\frac{1}{(n+1)^k}\int_{\mathbb{R}^2}\mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{z,n}\right]dz\right)^{\frac{1}{2}} \\
\le& C\|\nabla\varphi\|_{L^2}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\,l_b\sqrt{p_{k-2}(M)}
\end{aligned} \tag{4.13}$$

Step 2, high Landau levels. We remark that

$$\left|\mathrm{Tr}\left[\varphi\Pi_{>M}\gamma_b\right]\right| \le \sqrt{\mathrm{Tr}\left[\gamma_b|\varphi|^2\right]\mathrm{Tr}\left[\gamma_b\Pi_{>M}\right]} \le \|\varphi\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\Pi_{>M}\right]}$$

and

$$\mathrm{Tr}\left[\gamma_b \Pi_{>M}\right] \leq \sum_{n>M} \frac{n^k}{M^k} \mathrm{Tr}\left[\gamma_b \Pi_n\right] \leq \frac{1}{M^k} \sum_{n>M} \mathrm{Tr}\left[\gamma_b \Pi_n \mathscr{L}_b^k\right] = \frac{1}{M^k} \mathrm{Tr}\left[\gamma_b \Pi_{>M} \mathscr{L}_b^k\right]. \tag{4.14}$$

Hence

$$\left|\mathrm{Tr}\left[\varphi \Pi_{>M} \gamma_b\right]\right| \leq \|\varphi\|_{L^\infty} M^{-\frac{k}{2}} \sqrt{\mathrm{Tr}\left[\gamma_b \Pi_{>M} \mathscr{L}_b^k\right]} \tag{4.15}$$

We conclude by combining (4.13), (4.15) and (4.10). □

4.2. **Improved convergence with confinement.** To complete the proof of Theorem 2.4 we need to improve (4.4) to an estimate in Wasserstein-1 distance. We may do this at the price of an extra confining assumption:

Proposition 4.4 (**Convergence of the truncated semi-classical density with confinement**).
Let $\beta > 0, p > 3$. *We make the same assumptions as in Proposition 4.2, with in addition*

$$\mathrm{Tr}\left[\gamma_b |X|^p\right] < \infty \tag{4.16}$$

If $4\sqrt{M} l_b \leq l_b^{-\beta}$, *then*

$$W_1\left(\rho_{\gamma_b}, \rho_{\gamma_b}^{sc,\leq M}\right) \leq C(p)\left(1 + \mathrm{Tr}\left[\gamma_b |X|^p\right]\right)\left(\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b^{1-\beta} \begin{cases} M^{1-\frac{k}{2}} & \text{if } k<2 \\ \sqrt{\ln(M)} & \text{if } k=2 \\ 1 & \text{if } k>2 \end{cases} \right.$$
$$\left. + M^{-\frac{k}{2}} \sqrt{\mathrm{Tr}\left[\gamma_b \Pi_{>M} \mathscr{L}_b^k\right]} + l_b^{\beta(p-1)} + M l_b^{\beta(p-3)-2}\right)$$

We need a technical lemma containing some basic estimates:

Lemma 4.5 (**Technical integration results**).
Recalling the definition (3.7),

$$\left\|\psi_{z,n}\right\|_{L^1} \leq C\,(n+1)^{\frac{1}{4}}\, l_b. \tag{4.17}$$

Let $n \in \mathbb{N}, a > 0$ *and*

$$I_n(a) := \int_a^\infty t^n e^{-\frac{t^2}{2}} dt$$

then

$$I_{2n+1}(a) = 2^n n! e^{-\frac{a^2}{2}} \sum_{i=0}^n \frac{1}{i!}\left(\frac{a^2}{2}\right)^i.$$

Proof. Using Stirling's formula

$$\begin{aligned}
&\left\|\Psi_{z,n}\right\|_{L^1}\\
=&\frac{1}{\sqrt{2\pi n!}l_b}\int_{\mathbb{R}^2}\left|\frac{x-z}{\sqrt{2l_b}}\right|^n e^{-\frac{|x-z|^2}{4l_b^2}}dx=\sqrt{\frac{2}{\pi n!}}l_b\int_{\mathbb{R}^2}|x|^n e^{-\frac{|x|^2}{2}}dx=\sqrt{\frac{2\pi}{n!}}l_b\int_{\mathbb{R}_+}2t^{n+1}e^{-\frac{t^2}{2}}dt\\
=&\sqrt{\frac{2\pi}{n!}}l_b\int_{\mathbb{R}_+}t^{\frac{n}{2}}e^{-\frac{t}{2}}dt=\sqrt{\frac{2\pi}{n!}}l_b2^{\frac{n}{2}+1}\int_{\mathbb{R}_+}t^{\frac{n}{2}}e^{-t}dt=2^{\frac{n+3}{2}}\sqrt{\pi}l_b\frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{n!}}\\
\underset{n\to\infty}{\sim}&2^{\frac{n+3}{2}}\sqrt{\pi}l_b\frac{\sqrt{2\pi\frac{n}{2}}\left(\frac{n}{2e}\right)^{\frac{n}{2}}}{(2\pi n)^{\frac14}\left(\frac{n}{e}\right)^{\frac{n}{2}}}=2^{\frac54}\pi^{\frac34}n^{\frac14}l_b\underset{n\to\infty}{\sim}2^{\frac54}\pi^{\frac34}(n+1)^{\frac14}l_b.
\end{aligned}$$

As for the second claim in the lemma, an integration by parts shows

$$I_{n+2}(a)=(n+1)\,I_n(a)+a^{n+1}e^{-\frac{a^2}{2}}$$

so for odd integers

$$I_{2(n+1)+1}(a)=I_{2n+1+2}(a)=2\,(n+1)\,I_{2n+1}(a)+a^{2(n+1)}e^{-\frac{a^2}{2}}$$

hence by induction,

$$I_{2n+1}(a)=2^n n!\left(I_1(a)+e^{-\frac{a^2}{2}}\sum_{i=1}^{n}\frac{1}{i!}\left(\frac{a^2}{2}\right)^i\right)$$

and

$$I_1(a)=e^{-\frac{a^2}{2}}$$

corresponds to the index $i=0$ in the sum. □

We now turn to the

Proof of Proposition 4.4. Let φ a Lipschitz function. Since

$$\int_{\mathbb{R}^2}\varphi\left(\rho_{\gamma_b}-\rho_{\gamma_b}^{sc,\le M}\right)=\int_{\mathbb{R}^2}(\varphi-\varphi(0))\left(\rho_{\gamma_b}-\rho_{\gamma_b}^{sc,\le M}\right)$$

we can assume without loss of generality that $\varphi(0)=0$, and hence that

$$|\varphi(x)|\le\|\nabla\varphi\|_{L^\infty}|x| \tag{4.18}$$

Let

$$R:=l_b^{-\beta}$$

and define a partition of unity , $\chi_R,\eta_R\in C^\infty\left(\mathbb{R}^2,\mathbb{R}_+\right)$ such that

$$\begin{aligned}
&\chi_R+\eta_R=1\\
&|z|\le R\implies\chi_R=1,\eta_R=0\\
&|z|\ge 2R\implies\chi_R=0,\eta_R=1\\
&\left\|\nabla\eta_R\right\|_{L^\infty}\le\frac{2}{R}
\end{aligned} \tag{4.19}$$

Using (4.10) for $\varphi\chi_R$ instead of φ, we decompose

$$\int_{\mathbb{R}^2} \varphi \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leq M}\right) = \frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \mathrm{Tr}\left[\left(\varphi\chi_R - \varphi(z)\chi_R(z)\right)\Pi_{z,\leq M}\gamma_b\right] dz + \mathrm{Tr}\left[\varphi\chi_R\Pi_{>M}\gamma_b\right] + \int_{\mathbb{R}^2} \varphi\eta_R \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leq M}\right) \tag{4.20}$$

Step 1, low Landau levels. With the bounds (4.18) and (4.19) we find that, $\forall\, |z| \leq 2R$,

$$\left|\nabla\left(\varphi\chi_R\right)(z)\right| \leq |\nabla\varphi(z)| + \frac{2}{R}|\varphi(z)| \leq |\nabla\varphi(z)| + \frac{2}{R}|z|\,\|\nabla\varphi\|_{L^\infty} \leq 5\,\|\nabla\varphi\|_{L^\infty}$$

Hence

$$\left\|\nabla\left(\varphi\chi_R\right)(z)\right\|_{L^2} \leq 5\,\|\nabla\varphi\|_{L^\infty}\sqrt{|B(0,2R)|} \leq C\,\|\nabla\varphi\|_{L^\infty} R$$

Using (4.13) we deduce

$$\left|\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2} \mathrm{Tr}\left[\left(\varphi\chi_R - \varphi(z)\chi_R(z)\right)\gamma_b\Pi_{z,\leq M}\right] dz\right| \leq CRl_b\,\|\nabla\varphi\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\sqrt{p_{k-2}(M)} \tag{4.21}$$

Step 2, high Landau levels. Since

$$\left|\varphi(z)\chi_R(z)\right| \leq \|\nabla\varphi\|_{L^\infty}|z|$$

it follows from (4.18) that

$$\left|\mathrm{Tr}\left[\varphi\chi_R\Pi_{>M}\gamma_b\right]\right| \leq \sqrt{\mathrm{Tr}\left[\gamma_b\left|\varphi\chi_R\right|^2\right]\mathrm{Tr}\left[\gamma_b\Pi_{>M}\right]} \leq \|\nabla\varphi\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b|X|^2\right]}\sqrt{\mathrm{Tr}\left[\gamma_b\Pi_{>M}\right]}.$$

Inserting (4.14), we deduce

$$\left|\mathrm{Tr}\left[\varphi\chi_R\Pi_{>M}\gamma_b\right]\right| \leq \|\nabla\varphi\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b|X|^2\right]}M^{-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\Pi_{>M}\mathscr{L}_b^k\right]} \tag{4.22}$$

Step 3, tails of the densities. We next estimate the third term in (4.20) . First, using (4.18),

$$\left|\int_{\mathbb{R}^2}\varphi\eta_R\rho_{\gamma_b}\right| \leq \|\nabla\varphi\|_{L^\infty}\int_{z\geq R}|z|\,\rho_{\gamma_b}(z)dz \leq \frac{\|\nabla\varphi\|_{L^\infty}}{R^{p-1}}\int_{|z|\geq R}|z|^p\rho_{\gamma_b}(z)dz \leq \frac{\|\nabla\varphi\|_{L^\infty}}{R^{p-1}}\mathrm{Tr}\left[\gamma_b|X|^p\right]. \tag{4.23}$$

On the other hand, using (4.18) again,

$$\left|\int_{\mathbb{R}^2}\varphi\eta_R\rho_{\gamma_b}^{sc,\leq M}\right| \leq \|\nabla\varphi\|_{L^\infty}\int_{|z|\geq R}|z|\,\rho_{\gamma_b}^{sc,\leq M}(z)dz = \frac{\|\nabla\varphi\|_{L^\infty}}{2\pi l_b^2}\sum_{n=0}^{M}\int_{|z|\geq R}|z|\,\mathrm{Tr}\left[\gamma_b\Pi_{z,n}\right]dz \tag{4.24}$$

We write the spectral decomposition of γ_b in the manner

$$\gamma_b =: \sum_{i\in\mathbb{N}}\lambda_i\,|u_i\rangle\langle u_i|$$

with $0 \le \lambda_i \le (2\pi)^{-1} l_b^{-2}$ and $(u_i)_i$ an orthonormal basis of L^2. For $|z| \ge R, n \le M$, using the Cauchy-Schwarz inequality, we estimate

$$\begin{aligned}\mathrm{Tr}\left[\gamma_b \Pi_{z,n}\right] &= \sum_{i\in\mathbb{N}} \lambda_i \left|\left\langle \psi_{z,n} | u_i \right\rangle\right|^2 \le \sum_{i\in\mathbb{N}} \lambda_i \left(\int_{\mathbb{R}^2} |u_i| \sqrt{|\psi_{z,n}|} \cdot \sqrt{|\psi_{z,n}|} \right)^2 \\ &\le \left\|\psi_{z,n}\right\|_{L^1} \sum_{i\in\mathbb{N}} \lambda_i \int_{\mathbb{R}^2} |u_i|^2 \left|\psi_{z,n}\right| = \left\|\psi_{z,n}\right\|_{L^1} \int_{\mathbb{R}^2} \rho_{\gamma_b} \left|\psi_{z,n}\right| \end{aligned} \tag{4.25}$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^2} \rho_{\gamma_b} \left|\psi_{z,n}\right| &= \int_{\mathbb{R}^2} \rho_{\gamma_b}(x) \left(1 + |x|^p\right) \frac{\left|\psi_{z,n}(x)\right|}{1+|x|^p} dx \le \mathrm{Tr}\left[\gamma_b \left(1+|X|^p\right)\right] \left\| \frac{\psi_{n,z}}{1+|\bullet|^p} \right\|_{L^\infty} \\ &= \frac{\left(1+\mathrm{Tr}\left[\gamma_b |X|^p\right]\right)}{\sqrt{2\pi n!} l_b} \sup_{x\in\mathbb{R}^2} \frac{|x|^n e^{-\frac{|x|^2}{2}}}{1+\left|z+\sqrt{2} l_b x\right|^p} \end{aligned} \tag{4.26}$$

The function $t \mapsto t^n e^{-\frac{t^2}{2}}$ attains its' global maximal on $\mathbb{R}^+$ at $t = \sqrt{n}$ with maximal value $\left(\frac{n}{e}\right)^{\frac{n}{2}}$ and is decreasing for $t > \sqrt{n}$.

Since we assume $4\sqrt{M} l_b \le l_b^{-\beta}$, if $4 l_b |x| \ge |z|$

$$|x| \ge \frac{|z|}{4 l_b} \ge \frac{R}{4 l_b} \ge \sqrt{M} \ge \sqrt{n}$$

for b large enough, so in this case

$$\frac{|x|^n e^{-\frac{|x|^2}{2}}}{1+\left|z+\sqrt{2} l_b x\right|^p} \le \left(4\frac{|z|}{l_b}\right)^n e^{-\frac{1}{2}\left(4\frac{|z|}{l_b}\right)^2} \tag{4.27}$$

If instead $4 l_b |x| \le |z|$, we have

$$\left|z+\sqrt{2} l_b x\right| \ge |z| - \sqrt{2} l_b |x| \ge |z| \left(1 - \frac{\sqrt{2}}{4}\right) \ge \frac{|z|}{2}$$

and thus

$$\frac{|x|^n e^{-\frac{|x|^2}{2}}}{1+\left|z+\sqrt{2} l_b x\right|^p} \le \frac{\left(\frac{n}{e}\right)^{\frac{n}{2}}}{1+\left|\frac{z}{2}\right|^p} \tag{4.28}$$

Putting (4.27) (4.28) and (4.26) together leads to

$$\int_{\mathbb{R}^2} \rho_{\gamma_b} \left|\psi_{z,n}\right| \le \frac{\left(1+\mathrm{Tr}\left[\gamma_b |X|^p\right]\right)}{\sqrt{2\pi n!} l_b} \left(\left(4\frac{|z|}{l_b}\right)^n e^{-\frac{1}{2}\left(4\frac{|z|}{l_b}\right)^2} + \frac{\left(\frac{n}{e}\right)^{\frac{n}{2}}}{1+\left|\frac{z}{2}\right|^p} \right)$$

Inserting this and (4.17) in (4.25), combining with Stirling's formula again gives

$$\begin{aligned}\mathrm{Tr}\left[\gamma_b\Pi_{z,n}\right] \leq & C\left(1+\mathrm{Tr}\left[\gamma_b\left|X\right|^p\right]\right)\frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}}\left(\left(4\frac{|z|}{l_b}\right)^n e^{-\frac{1}{2}\left(4\frac{|z|}{l_b}\right)^2}+\frac{\left(\frac{n}{e}\right)^{\frac{n}{2}}}{1+\left|\frac{z}{2}\right|^p}\right)\\ \leq & C\left(1+\mathrm{Tr}\left[\gamma_b\left|X\right|^p\right]\right)\left(\frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}}\left(4\frac{|z|}{l_b}\right)^n e^{-\frac{1}{2}\left(4\frac{|z|}{l_b}\right)^2}+\frac{1}{1+\left|\frac{z}{2}\right|^p}\right)\end{aligned}$$

With the notation of Lemma 4.5,

$$\int_{|z|\geq R}|z|\left(4\frac{|z|}{l_b}\right)^n e^{-\frac{1}{2}\left(4\frac{|z|}{l_b}\right)^2}dz=\frac{\pi}{32}l_b^3\int_{\frac{4R}{l_b}}^{\infty}r^{n+2}e^{-\frac{1}{2}r^2}=\frac{\pi}{32}l_b^3 I_{n+2}\left(\frac{4R}{l_b}\right)$$

Since $p>3$, we can integrate

$$\int_{|z|\geq R}\frac{|z|}{1+\left|\frac{z}{2}\right|^p}dz\leq 2^p\int_{|z|\geq R}|z|^{1-p}\,dz=2^{p+1}\pi\int_R^{\infty}r^{2-p}dr=\frac{2^{p+1}\pi}{(p-3)\,R^{p-3}}$$

Collecting the above considerations gives

$$\int_{|z|\geq R}|z|\,\mathrm{Tr}\left[\gamma_b\Pi_{z,n}\right]dz\leq C(p)\left(1+\mathrm{Tr}\left[\gamma_b\left|X\right|^p\right]\right)\left(\frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}}l_b^3 I_{n+2}\left(\frac{4R}{l_b}\right)+\frac{1}{R^{p-3}}\right)$$

that we may combine with (4.24) to get

$$\left|\int_{\mathbb{R}^2}\varphi\eta_R\rho_{\gamma_b}^{sc,\leq M}\right|\leq\frac{\|\nabla\varphi\|_{L^\infty}}{2\pi l_b^2}\sum_{n=0}^{M}\int_{|z|\geq R}|z|\,\mathrm{Tr}\left[\gamma_b\Pi_{z,n}\right]dz \tag{4.29}$$

$$\leq C(p)\,\|\nabla\varphi\|_{L^\infty}\left(1+\mathrm{Tr}\left[\gamma_b\left|X\right|^p\right]\right)\left(l_b\sum_{n=0}^{M}\frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}}I_{n+2}\left(\frac{4R}{l_b}\right)+\frac{M}{l_b^2R^{p-3}}\right) \tag{4.30}$$

But $R\geq 4\sqrt{M}l_b\gg l_b$ so $\left(I_n\left(\frac{R}{4l_b}\right)\right)_{n\in\mathbb{N}}$ is increasing, hence it follows from Lemma 4.5 that

$$\begin{aligned}\sum_{n=0}^{M}\frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}}I_{n+2}\left(\frac{4R}{l_b}\right)\leq & \sum_{n=0}^{2\left\lfloor\frac{M}{2}\right\rfloor+1}\frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}}I_{n+2}\left(\frac{4R}{l_b}\right)\\ = & \sum_{n=0}^{\left\lfloor\frac{M}{2}\right\rfloor}\frac{(2n+1)^{\frac{1}{4}}}{\sqrt{(2n)!}}I_{2n+2}\left(\frac{4R}{l_b}\right)+\sum_{n=0}^{\left\lfloor\frac{M}{2}\right\rfloor}\frac{(2n+2)^{\frac{1}{4}}}{\sqrt{(2n+1)!}}I_{2n+3}\left(\frac{4R}{l_b}\right)\\ \leq & 2\sum_{n=0}^{\left\lfloor\frac{M}{2}\right\rfloor}\frac{(2n+2)^{\frac{1}{4}}}{\sqrt{(2n)!}}I_{2n+3}\left(\frac{4R}{l_b}\right)\\ = & 2e^{-8\frac{R^2}{l_b^2}}\sum_{n=0}^{\left\lfloor\frac{M}{2}\right\rfloor}\frac{(2n+2)^{\frac{1}{4}}}{\sqrt{(2n)!}}2^{n+1}\,(n+1)!\sum_{i=0}^{n+1}\frac{1}{i!}\left(8\frac{R^2}{l_b^2}\right)^i\end{aligned}$$

Then using $(2n)! \geq (n!)^2, i! \geq 1$ and $8\frac{R^2}{l_b^2} \geq 1$,

$$\begin{aligned}
&\sum_{n=0}^{M} \frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} I_{n+2}\left(\frac{4R}{l_b}\right) \\
\leq &2e^{-8\frac{R^2}{l_b^2}} \sum_{n=0}^{\left\lfloor \frac{M}{2} \right\rfloor} (2n+2)^{\frac{1}{4}} (n+1)^2 2^{n+1} \left(8\frac{R^2}{l_b^2}\right)^{n+1} \\
\leq &2e^{-8\frac{R^2}{l_b^2}} \left(\left\lfloor \frac{M}{2} \right\rfloor + 1\right)\left(2\left\lfloor \frac{M}{2} \right\rfloor + 2\right)^{\frac{1}{4}} \left(\left\lfloor \frac{M}{2} \right\rfloor + 1\right)^2 2^{\left\lfloor \frac{M}{2} \right\rfloor + 1} \left(8\frac{R^2}{l_b^2}\right)^{\left\lfloor \frac{M}{2} \right\rfloor + 1} \\
\leq &CM^{\frac{13}{4}} 4^M \left(\frac{R^2}{l_b^2}\right)^{\frac{M}{2}+1} e^{-8\frac{R^2}{l_b^2}} = CM^{\frac{13}{4}} e^{(1+\beta)(M+2)\ln\left(l_b^{-1}\right) + M\ln(4) - 8l_b^{-2(1+\beta)}}
\end{aligned}$$

Since $M \ll l_b^{-2}$ for large enough b,

$$\sum_{n=0}^{M} \frac{(n+1)^{\frac{1}{4}}}{\sqrt{n!}} I_{n+2}\left(\frac{4R}{l_b}\right) \leq Cl_b^{-\frac{13}{2}} e^{l_b^{-2}\ln\left(l_b^{-1}\right) - 8l_b^{-2(1+\beta)}} = Cl_b^{-\frac{13}{2}} e^{-l_b^{-2(1+\beta)}\left(8 - l_b^{2\beta}\ln\left(l_b^{-1}\right)\right)}.$$

Inserting this in (4.29), we get

$$\left|\int_{\mathbb{R}^2} \varphi \eta_R \rho_{\gamma_b}^{sc,\leq M}\right| \leq C(p) \|\nabla\varphi\|_{L^\infty} \left(1 + \operatorname{Tr}\left[\gamma_b |X|^p\right]\right) \left(l_b^{-\frac{11}{2}} e^{-l_b^{-2(1+\beta)}\left(8 - l_b^{2\beta}\ln\left(l_b^{-1}\right)\right)} + \frac{M}{l_b^2 R^{p-3}}\right) \tag{4.31}$$

Step 4, conclusion. We insert (4.21) (4.22) (4.23) (4.31) in (4.20):

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leq M}\right)\right| \leq C(p) \|\nabla\varphi\|_{L^\infty} \left(\sqrt{\operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b^{1-\beta} \sqrt{p_{k-2}(M)}\right. \\
&+ \sqrt{\operatorname{Tr}\left[\gamma_b |X|^2\right]} M^{-\frac{k}{2}} \sqrt{\operatorname{Tr}\left[\gamma_b \Pi_{>M} \mathscr{L}_b^k\right]} + \operatorname{Tr}\left[\gamma_b |X|^p\right] l_b^{\beta(p-1)} \\
&\left. + \left(1 + \operatorname{Tr}\left[\gamma_b |X|^p\right]\right)\left(l_b^{-\frac{11}{2}} e^{-l_b^{-2(1+\beta)}\left(8 - l_b^{2\beta}\ln\left(l_b^{-1}\right)\right)} + M l_b^{\beta(p-3)-2}\right)\right)
\end{aligned}$$

Since $p > 3$, we conclude by noting that

$$\sqrt{\operatorname{Tr}\left[\gamma_b |X|^2\right]} \leq \sqrt{1 + \operatorname{Tr}\left[\gamma_b |X|^p\right]} \leq 1 + \operatorname{Tr}\left[\gamma_b |X|^p\right].$$

□

5. Semi-classical dynamics of the densities

This section contains the proof of Theorem 2.3. We first provide the second main ingredient mentioned after the statement, namely the study of the dynamics of the truncated semi-classical density. We next combine this with the bounds of Section 4 to conclude the proof.

5.1. **Dynamics of the truncated semi-classical density.** We now prove that $\rho_{\gamma_b}^{sc,\leq M}$, as defined in Section 4, almost satisfies the weak formulation of (2.5) modulo a suitable choice of $1 \ll M \ll l_b^{-2}$. Here W is a (possibly time-dependent) generic potential that we will replace with $V + w \star \rho_{\gamma_b(t)}$ later on.

Proposition 5.1 (**Drift equation for the truncated semi-classical density**).
Let $t \in \mathbb{R}_+, k \geq 0, l \geq 1, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{l+1,\infty}(\mathbb{R}^2)$ *and assume*

$$\begin{aligned}&\operatorname{Tr}\left[\gamma_b(t)\right] = 1, 0 \leq \gamma_b(t) \leq 2\pi l_b^2\\&\operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] < \infty\\&\partial_t\gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right].\end{aligned}$$

Then, $\forall\varphi \in L^1 \cap W^{1,\infty}\left(\mathbb{R}^2\right)$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\varphi(z)\left(\partial_t\rho_{\gamma_b}^{sc,\leq M}(t,z) + \nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc,\leq M}(t,z)\right)dz\right|\\&\leq C(l)\left\|\varphi\right\|_{L^1\cap W^{1,\infty}}\left\|W\right\|_{W^{l+1,\infty}}\left(\sqrt{\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\,l_b^{l-1}\begin{cases}M^{1+\frac{l-k}{2}} & \text{if } k < l+2\\ \sqrt{\ln(M)} & \text{if } k = l+2\\ 1 & \text{if } k > l+2\end{cases}\right.\\&\left.+\frac{\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{M-l+1:M+l}\right]}{l_b M^{k-\frac{1}{2}}} + \operatorname{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\sum_{p=2}^{l} l_b^{p-1}\begin{cases}M^{\frac{p+1}{2}-k} & \text{if } k < \frac{p+1}{2}\\ \ln(M) & \text{if } k = \frac{p+1}{2}\\ 1 & \text{if } k > \frac{p+1}{2}\end{cases}\right)\end{aligned}$$

The proof will proceed from a Taylor expansion of the potential W at order l. Several bounds will be proved by induction on l and are not particularly more complicated to write in the above generality. However our choice of l will be set by the a priori bound available to us, as we explain first:

Remark 5.2 (**Choice of the expansion parameters**).
In the proof of Theorem 2.3 we take $k = 1$ to be able to use Lemma 3.9. Then Proposition 5.1 gives

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\varphi\left(\partial_t\rho_{\gamma_b}^{sc,\leq M}(t) + \nabla^\perp W\cdot\nabla\rho_{\gamma_b}^{sc,\leq M}(t)\right)\right| \leq C(l)\left\|\varphi\right\|_{L^1\cap W^{1,\infty}}\left\|W\right\|_{W^{l+1,\infty}}\\&\left(\sqrt{\operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]}\,l_b^{l-1}M^{\frac{l+1}{2}} + \frac{\operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{M-l+1:M+l}\right]}{l_b\sqrt{M}} + \operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]l_b\sqrt{M}\right) \qquad (5.1)\end{aligned}$$

The first error above is of order $l_b^{l-1}M^{\frac{l+1}{2}}$, the third is of order $l_b\sqrt{M}$. In Lemma 5.6 we will see that $M \gg 1$ may be chosen so that the second error weighs $l_b^{-1}M^{-\frac{3}{2}}$. We are able to control all the errors if

$$\left(\frac{1}{l_b}\right)^{\frac{2}{3}} \ll M \ll \left(\frac{1}{l_b}\right)^{2\frac{l-1}{l+1}}$$

which is possible if and only if $l > 2$. Hence the above will be applied to obtain Theorem 2.3 with the choices $k = 1, l = 3$. This will yield error terms

$$l_b^2M^2 + l_b^{-1}M^{-\frac{3}{2}} + l_bM^{1/2}.$$

Since the second term must be $o(1)$, the first one dominates the third, and optimization in M leads to a choice $M \sim l_b^{-6/7}$ and a final error of order $O(l_b^{2/7})$.

If more moments of the kinetic energy are bounded for positive times, we may use the above with a larger value of k and hence get an efficient estimate for lower l, i.e. asking for less regularity of the external and interaction potentials.

$\diamond$

The first step in our proof consists in deriving an explicit equation satisfied by $\rho^{sc,\le M}_{\gamma_b}$. Most bounds are then obtained by writing quantum expectations (traces) using integrals of operator kernels. A general term will be an integral in $x, y, z \in \mathbb{R}^2$ where x, y are the arguments of the operator kernel and z the guiding-center coordinate of the coherent state at hand. As per the observations of Section 3.2 such integrals are concentrated around $x \sim y \sim z$. Principal terms corresponding to Equation (2.5) are easily identified by replacing $W(x) - W(y)$ (coming from the commutator with W) by $(x-y)\cdot\nabla W(z)$ and then using (3.12). Some care in controling the expansion of W is needed:

• Using (3.9) (or more precisely, the first term in (3.11)) a factor of $(x - y)$ can make us gain[2] a factor of l_b^2, at the price of integrations by parts (that we can perform using the regularity of the test function).

• A factor of $|x - z|$ or $|y - z|$ makes us gain at best a factor of $l_b\sqrt{n}$, the localization length of the coherent state wave-function in the n-th Landau level. Cf the discussion below Proposition 4.2.

• There is a truncation error due to the second term in (3.9), calling for some optimization in M.

In (5.5) below the remainder is in a form allowing to leverage the first observation above. Indeed, the commutator naturally brings factors of $(x-y)$ (think of the commutator with the position operator X). The remainders in (5.6) are estimated using the second observation, which we formalize in Lemma 5.4 below. Finally (5.7) is the truncation error, whose control will demand a proper choice of M later in the proof, see Lemma 5.6.

Lemma 5.3 (**Equation for the semi-classical density and potential expansion**).
Let $t \in \mathbb{R}_+, l \in \mathbb{N}, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{l,\infty}(\mathbb{R}^2)$ *and assume*

$$\partial_t\gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right]. \tag{5.2}$$

Denote d^pW_z *the* p^{th} *differential of* W *at* $z \in \mathbb{R}^2$*, meaning that* d^pW_z *is a* p*-linear form on* $\mathbb{R}^2$*.*

Then, with $\rho^{sc,\le M}_{\gamma_b}$ *as in Definition 4.1,*

$$\partial_t\rho^{sc,\le M}_{\gamma_b} + \nabla^\perp W\cdot\nabla\rho^{sc,\le M}_{\gamma_b} = \mathcal{E}^l_0 + \sum_{p=2}^{l}\sum_{q=1}^{p}\mathcal{E}^{p,q}_{\mathrm{I}} + \sum_{p=1}^{l}\sum_{q=1}^{p}\mathcal{E}^{p,q}_{\mathrm{II}} \tag{5.3}$$

where

$$\mathcal{V}_{z,l}(x) := W(x) - \sum_{p=0}^{l}\frac{1}{p!}d^pW_z\,(x-z)^{\otimes p} \tag{5.4}$$

$$\mathcal{E}^l_0(z) := \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{z,\le M}, \mathcal{V}_{z,l}\right]\right] \tag{5.5}$$

[2]Note that this aspect is algebraic in nature: it is not clear that a factor of $|x - y|$ would gain us a factor of l_b^2.

and

$$\mathcal{E}_{\mathrm{I}}^{p,q}(z) := \frac{1}{2\pi l_b^2 p!}\mathrm{Tr}\left[\gamma_b d^p W_z (X-z)^{\otimes(q-1)} \otimes \left(\nabla_z^\perp \Pi_{z,\leq M}\right) \otimes (X-z)^{\otimes(p-q)}\right] \tag{5.6}$$

$$\mathcal{E}_{\mathrm{II}}^{p,q}(z) := \frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3 p!}$$
$$\mathrm{Tr}\left[\gamma_b d^p W_z (X-z)^{\otimes(q-1)} \otimes \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \left|\psi_{z,M}\right\rangle\left\langle\psi_{z,M+1}\right| \\ \left|\psi_{z,M+1}\right\rangle\left\langle\psi_{z,M}\right| \end{pmatrix} \otimes (X-z)^{\otimes(p-q)}\right] \tag{5.7}$$

Proof. **Step 1 : Direct computation.** We start from (4.2) and Equation (5.2):

$$\begin{aligned}\partial_t \rho_{\gamma_b}^{sc,\leq M}(z) =& \frac{1}{2\pi l_b^2}\mathrm{Tr}\left[\partial_t\gamma_b \Pi_{z,\leq M}\right] = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\left[\mathscr{L}_b + W, \gamma_b\right]\Pi_{z,\leq M}\right] \\ =& \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{z,\leq M}, \mathscr{L}_b + W\right]\right] = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{z,\leq M}, W\right]\right]\end{aligned} \tag{5.8}$$

On the other hand, using (3.12),

$$\begin{aligned}\nabla^\perp W(z)\cdot\nabla_z \rho_{\gamma_b}^{sc,\leq M}(z) &= -\nabla W(z)\cdot\nabla_z^\perp \rho_{\gamma_b}^{sc,\leq M}(z) = \frac{-1}{2\pi l_b^2}\nabla W(z)\cdot\mathrm{Tr}\left[\gamma_b \nabla_z^\perp \Pi_{z,\leq M}\right] \\ &= \frac{-1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{z,\leq M}, X\cdot\nabla W(z)\right]\right] \\ &\quad + \frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3}\nabla W(z)\cdot\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \mathrm{Tr}\left[\gamma_b\left|\psi_{z,M}\right\rangle\left\langle\psi_{z,M+1}\right|\right] \\ \mathrm{Tr}\left[\gamma_b\left|\psi_{z,M+1}\right\rangle\left\langle\psi_{z,M}\right|\right]\end{pmatrix}\end{aligned} \tag{5.9}$$

Putting together Equation (5.8) and Equation (5.9) yields

$$\begin{aligned}\partial_t \rho_{\gamma_b}^{sc,\leq M}(z) + \nabla^\perp W(z)\cdot\nabla_z \rho_{\gamma_b}^{sc,\leq M}(z) = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{z,\leq M}, W - X\cdot\nabla W(z)\right]\right] \\ + \frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3}\nabla W(z)\cdot\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \mathrm{Tr}\left[\gamma_b\left|\psi_{z,n}\right\rangle\left\langle\psi_{z,M+1}\right|\right] \\ \mathrm{Tr}\left[\gamma_b\left|\psi_{z,M+1}\right\rangle\left\langle\psi_{z,n}\right|\right]\end{pmatrix}\end{aligned} \tag{5.10}$$

Step 2 : Taylor expansion. We compute

$$\begin{aligned}\mathcal{V}_{z,l}(y) - \mathcal{V}_{z,l}(x) =& W(y) - W(x) - \sum_{p=0}^{l}\frac{1}{p!}d^p W_z\left((y-z)^{\otimes p} - (x-z)^{\otimes p}\right) \\ =& W(y) - W(x) - (y-x)\cdot\nabla W(z) - \sum_{p=2}^{l}\frac{1}{p!}d^p W_z\left((y-z)^{\otimes p} - (x-z)^{\otimes p}\right)\end{aligned}$$

and notice the telescopic expression

$$
\begin{aligned}
(y-z)^{\otimes p}-(x-z)^{\otimes p} &= \sum_{q=0}^{p-1}(x-z)^{\otimes q}\otimes(y-z)^{\otimes(p-q)}-\sum_{q=1}^{p}(x-z)^{\otimes q}\otimes(y-z)^{\otimes(p-q)}\\
&= \sum_{q=1}^{p}(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p+1-q)}-\sum_{q=1}^{p}(x-z)^{\otimes q}\otimes(y-z)^{\otimes(p-q)}\\
&= \sum_{q=1}^{p}(x-z)^{\otimes(q-1)}\otimes(y-z-(x-z))\otimes(y-z)^{\otimes(p-q)}\\
&= \sum_{q=1}^{p}(x-z)^{\otimes(q-1)}\otimes(y-x)\otimes(y-z)^{\otimes(p-q)}
\end{aligned}
$$

Combining the above we express the integral kernel

$$
\begin{aligned}
&\left[\Pi_{z,\le M},W-X\cdot\nabla W(z)\right](x,y)=\Pi_{z,\le M}(x,y)\left(W(y)-W(x)-(y-x)\cdot\nabla W(z)\right)\\
=&\Pi_{z,\le M}(x,y)\left(\mathcal{V}_{z,l}(y)-\mathcal{V}_{z,l}(x)+\sum_{p=2}^{l}\frac{1}{p!}d^pW_z\left((y-z)^{\otimes p}-(x-z)^{\otimes p}\right)\right)\\
=&\Pi_{z,\le M}(x,y)\left(\mathcal{V}_{z,l}(y)-\mathcal{V}_{z,l}(x)+\sum_{p=2}^{l}\sum_{q=1}^{p}\frac{1}{p!}d^pW_z\,(x-z)^{\otimes(q-1)}\otimes(y-x)\otimes(y-z)^{\otimes(p-q)}\right)
\end{aligned}
$$

Inserting (3.12) we obtain the operator equality

$$
\begin{aligned}
&\left[\Pi_{z,\le M},W-X\cdot\nabla W(z)\right]\\
&=\left[\Pi_{z,\le M},\mathcal{V}_{z,l}\right]+\sum_{p=2}^{l}\sum_{q=1}^{p}\frac{1}{p!}d^pW_z\,(X-z)^{\otimes(q-1)}\otimes\left[\Pi_{z,\le M},X\right]\otimes(X-z)^{\otimes(p-q)}\\
&=\left[\Pi_{z,\le M},\mathcal{V}_{z,l}\right]+il_b^2\sum_{p=2}^{l}\sum_{q=1}^{p}\frac{1}{p!}d^pW_z\,(X-z)^{\otimes(q-1)}\otimes\left(\nabla_z^{\perp}\Pi_{z,\le M}\right)\otimes(X-z)^{\otimes(p-q)}\\
&+il_b\sqrt{\frac{M+1}{2}}\sum_{p=2}^{l}\sum_{q=1}^{p}\frac{1}{p!}d^pW_z\,(X-z)^{\otimes(q-1)}\otimes\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\left|\psi_{z,M}\right\rangle\left\langle\psi_{z,M+1}\right|\\\left|\psi_{z,M+1}\right\rangle\left\langle\psi_{z,M}\right|\end{pmatrix}\otimes(X-z)^{\otimes(p-q)}
\end{aligned}
\tag{5.11}
$$

Step 3 : Conclusion. Inserting (5.11) in Equation (5.10) we find

$$
\begin{aligned}
&\partial_t\rho_{\gamma_b}^{sc,\le M}+\nabla^{\perp}W\cdot\nabla\rho_{\gamma_b}^{sc,\le M}=\frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{z,\le M},\mathcal{V}_{z,l}\right]\right]\\
&+\sum_{p=2}^{l}\sum_{q=1}^{p}\frac{1}{2\pi l_b^2p!}\mathrm{Tr}\left[\gamma_bd^pW_z\,(X-z)^{\otimes(q-1)}\otimes\left(\nabla_z^{\perp}\Pi_{z,\le M}\right)\otimes(X-z)^{\otimes(p-q)}\right]\\
&+\sum_{p=1}^{l}\sum_{q=1}^{p}\frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3p!}\\
&\cdot\mathrm{Tr}\left[\gamma_bd^pW_z\,(X-z)^{\otimes(q-1)}\otimes\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\left|\psi_{z,M}\right\rangle\left\langle\psi_{z,M+1}\right|\\\left|\psi_{z,M+1}\right\rangle\left\langle\psi_{z,M}\right|\end{pmatrix}\otimes(X-z)^{\otimes(p-q)}\right]
\end{aligned}
$$

Note that the high Landau level error term in Equation (5.10) is exactly the term indexed by $p = 1$ in the second sum above. Finally, defining the errors terms (5.5), (5.6), (5.7), we get the decomposition Equation (5.3). □

In the proof of Proposition 5.1 we will use repeatedly the following bound to dispatch the error terms in (5.6) and (5.7).

Lemma 5.4 (**A general error term**).
Let γ_b *be a non negative trace-class operator,* $X = (X_1, X_2)$ *the position operator,* $r, M \in \mathbb{N}$, $z = (z_1, z_2) \in \mathbb{R}^2$ *and* $\Pi_{z,M}$ *the coherent state projector* (3.3). *Define*

$$\mathcal{E}_{r,M}(z) := \sum_{i_1 \dots i_r \in \{1,2\}^r} \mathrm{Tr}\left[\gamma_b \prod_{j=1}^{r} (X_{i_j} - z_{i_j}) \Pi_{z,M} \prod_{j=1}^{r} (X_{i_j} - z_{i_j})\right]. \tag{5.12}$$

We have the bound

$$\mathcal{E}_{r,M}(z) \leq C(r) l_b^{2r} M^{r-k} \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k \Pi_{z,M-r:M+r}\right] \tag{5.13}$$

with $\Pi_{z,M-r:M+r}$ *as in Definition 3.4.*

We first note a technical bound

Lemma 5.5 (**Some integrals**).
Let $\alpha \in \mathbb{R}_+$, *then*

$$I_n(\alpha) := \frac{1}{\pi n!} \int_{\mathbb{R}^2} |x|^{\alpha+2n} e^{-|x|^2} dx \underset{n\to\infty}{\sim} n^{\frac{\alpha}{2}}$$

and $\exists C > 0$ *such that*

$$\forall n \in \mathbb{M}, I_n(\alpha) \leq C(n+1)^{\frac{\alpha}{2}}$$

Proof. With polar coordinates and the change of variable $x^2 \mapsto x$

$$\frac{1}{\pi} \int_{\mathbb{R}^2} |x|^{\alpha} e^{-|x|^2} dx = 2 \int_{\mathbb{R}_+} x^{\alpha+1} e^{-x^2} dx = \int_{\mathbb{R}_+} x^{\frac{\alpha}{2}} e^{-u} du = \Gamma\left(\frac{\alpha}{2} + 1\right)$$

where Γ is the Euler Gamma function,

$$\Gamma(z) := \int_{\mathbb{R}_+} t^{z-1} e^{-t} dt$$

We have the following equivalent for the Euler Gamma function (as a direct consequence of the Stirling formula)

$$\frac{\Gamma(n+x)}{\Gamma(n)} \underset{x\to\infty}{\sim} n^x.$$

Thus

$$I_n(\alpha) = \frac{1}{n!} \Gamma\left(n + \frac{\alpha}{2} + 1\right) = \frac{\Gamma\left(n + \frac{\alpha}{2} + 1\right)}{\Gamma(n+1)}$$

and then

$$I_n(\alpha) \underset{n\to\infty}{\sim} (n+1)^{\frac{\alpha}{2}} \underset{n\to\infty}{\sim} n^{\frac{\alpha}{2}}.$$

□

We now proceed to the

Proof of Lemma 5.4. We use an induction in r. Define

$$\eta_b(z) = \left(\prod_{j=1}^{r}(X_{i_j} - z_{i_j})\right)\gamma_b \prod_{j=1}^{r}(X_{i_j} - z_{i_j})$$

We have

$$\begin{aligned}
&\mathrm{Tr}\left[\eta_b(z)(X_1 - z_1)\Pi_{z,M}(X_1 - z_1)\right] + \mathrm{Tr}\left[\eta_b(z)(X_2 - z_2)\Pi_{z,M}(X_2 - z_2)\right] \\
=&\mathrm{Tr}\left[\eta_b(z)\,(\mathbf{X} - \mathbf{z})\,\Pi_{z,M}\left(\overline{\mathbf{X} - \mathbf{z}}\right)\right] + \mathrm{Tr}\left[\eta_b(z)\left(\overline{\mathbf{X} - \mathbf{z}}\right)\Pi_{z,M}\,(\mathbf{X} - \mathbf{z})\right] \\
=&\mathrm{Tr}\left[\eta_b(z)\,(\mathbf{X} - \mathbf{z})\,\Pi_{z,M}\left(\overline{\mathbf{X} - \mathbf{z}}\right)\right] + 2l_b^2(M+1)\mathrm{Tr}\left[\eta_b(z)\Pi_{z,M+1}\right]
\end{aligned} \tag{5.14}$$

In view of estimating the first term by induction, we insert the coherent state resolution of identity (3.5) twice:

$$\begin{aligned}
&\mathrm{Tr}\left[\eta_b(z)\,(\mathbf{X} - \mathbf{z})\,\Pi_{z,M}\left(\overline{\mathbf{X} - \mathbf{z}}\right)\right] \\
&= \frac{1}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{Tr}\left[\eta_b(z)\Pi_{x,k}\,(\mathbf{X} - \mathbf{z})\,\Pi_{z,M}\left(\overline{\mathbf{X} - \mathbf{z}}\right)\Pi_{y,l}\right] dxdy
\end{aligned}$$

Then this becomes

$$\begin{aligned}
&\frac{1}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{Tr}\left[\eta_b(z)\Pi_{x,k}\,(\mathbf{X} - \mathbf{x})\,\Pi_{z,M}\left(\overline{\mathbf{X} - \mathbf{y}}\right)\Pi_{y,l}\right] dxdy \\
+&\frac{1}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\mathbf{x} - \mathbf{z})\,\mathrm{Tr}\left[\eta_b(z)\Pi_{x,k}\Pi_{z,M}\left(\overline{\mathbf{X} - \mathbf{y}}\right)\Pi_{y,l}\right] dxdy \\
+&\frac{1}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\overline{\mathbf{y} - \mathbf{z}}\right)\mathrm{Tr}\left[\eta_b(z)\Pi_{x,k}\,(\mathbf{X} - \mathbf{x})\,\Pi_{z,M}\Pi_{y,l}\right] dxdy \\
+&\frac{1}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\mathbf{x} - \mathbf{z})\left(\overline{\mathbf{y} - \mathbf{z}}\right)\mathrm{Tr}\left[\eta_b(z)\Pi_{x,k}\Pi_{z,M}\Pi_{y,l}\right] dxdy
\end{aligned} \tag{5.15}$$

and next

$$\begin{aligned}
&\frac{2l_b^2}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \sqrt{k+1}\sqrt{l+1} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{Tr}\left[\eta_b(z)\left|\psi_{x,k}\right\rangle\left\langle\psi_{x,k+1}\right|\Pi_{z,M}\left|\psi_{y,l+1}\right\rangle\left\langle\psi_{y,l}\right|\right] dxdy \\
-&\frac{i\sqrt{2}l_b}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \sqrt{l+1} \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\mathbf{x} - \mathbf{z})\,\mathrm{Tr}\left[\eta_b(z)\Pi_{x,k}\Pi_{z,M}\left|\psi_{y,l+1}\right\rangle\left\langle\psi_{y,l}\right|\right] dxdy \\
+&\frac{i\sqrt{2}l_b}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \sqrt{k+1} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\overline{\mathbf{y} - \mathbf{z}}\right)\mathrm{Tr}\left[\eta_b(z)\left|\psi_{x,k}\right\rangle\left\langle\psi_{x,k+1}\right|\Pi_{z,M}\Pi_{y,l}\right] dxdy \\
+&\frac{1}{\left(2\pi l_b^2\right)^2} \sum_{k,l\in\mathbb{N}^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} (\mathbf{x} - \mathbf{z})\left(\overline{\mathbf{y} - \mathbf{z}}\right)\mathrm{Tr}\left[\eta_b(z)\Pi_{x,k}\Pi_{z,M}\Pi_{y,l}\right] dxdy
\end{aligned} \tag{5.16}$$

further rewritten as

$$\begin{aligned}
&\frac{2l_b^2 M}{\left(2\pi l_b^2\right)^2}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{Tr}\left[\eta_b(z)\left|\psi_{x,M-1}\right\rangle\left\langle\psi_{x,M}\right|\Pi_{z,M}\left|\psi_{y,M}\right\rangle\left\langle\psi_{y,M-1}\right|\right]dxdy\\
&-\frac{i\sqrt{2}l_b\sqrt{M}}{\left(2\pi l_b^2\right)^2}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}(\mathbf{x}-\mathbf{z})\,\mathrm{Tr}\left[\eta_b(z)\Pi_{x,M}\Pi_{z,M}\left|\psi_{y,M}\right\rangle\left\langle\psi_{y,M-1}\right|\right]dxdy\\
&+\frac{i\sqrt{2}l_b\sqrt{M}}{\left(2\pi l_b^2\right)^2}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\left(\overline{\mathbf{y}-\mathbf{z}}\right)\mathrm{Tr}\left[\eta_b(z)\left|\psi_{x,M-1}\right\rangle\left\langle\psi_{x,M}\right|\Pi_{z,M}\Pi_{y,M}\right]dxdy\\
&+\frac{1}{\left(2\pi l_b^2\right)^2}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}(\mathbf{x}-\mathbf{z})\left(\overline{\mathbf{y}-\mathbf{z}}\right)\mathrm{Tr}\left[\eta_b(z)\Pi_{x,M}\Pi_{z,M}\Pi_{y,M}\right]dxdy
\end{aligned}\tag{5.17}$$

Using (3.6) we remark that

$$\begin{aligned}
\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\left|\psi_{x,M-1}\right\rangle\left\langle\psi_{x,M}\right|dx &= \sum_{m\in\mathbb{N}}\left|\varphi_{M-1,m}\right\rangle\left\langle\varphi_{M,m}\right|\\
\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\left|\psi_{y,M}\right\rangle\left\langle\psi_{y,M-1}\right|dx &= \sum_{m\in\mathbb{N}}\left|\varphi_{M,m}\right\rangle\left\langle\varphi_{M-1,m}\right|\\
\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}x\Pi_{M,x}dx &= \sqrt{2}l_b\sum_{m\in\mathbb{N}}\sqrt{m+1}\left|\varphi_{M,m}\right\rangle\left\langle\varphi_{M,m+1}\right|\\
\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\Pi_{M,y}\bar{y}dy &= \sqrt{2}l_b\sum_{m\in\mathbb{N}}\sqrt{m+1}\left|\varphi_{M,m+1}\right\rangle\left\langle\varphi_{M,m}\right|
\end{aligned}$$

Hence, inserting

$$\begin{aligned}
&\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\left|\psi_{x,M-1}\right\rangle\left\langle\psi_{x,M}\right|dx\Pi_{z,M}\\
=&e^{-\frac{|z|^2}{4l_b^2}}\sum_{m\in\mathbb{N}}\frac{\left|\varphi_{M-1,m}\right\rangle\left\langle\psi_{z,M}\right|}{\sqrt{m!}}\left(\frac{z}{\sqrt{2}l_b}\right)^m = \left|\psi_{z,M-1}\right\rangle\left\langle\psi_{z,M}\right|\\
&\Pi_{z,M}\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\left|\psi_{y,M}\right\rangle\left\langle\psi_{y,M-1}\right|dx = \left|\psi_{z,M}\right\rangle\left\langle\psi_{z,M-1}\right|\\
&\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}x\Pi_{M,x}dx\Pi_{z,M} = \sqrt{2}l_b e^{-\frac{|z|^2}{4l_b^2}}\sum_{m\in\mathbb{N}}\frac{\sqrt{m+1}}{\sqrt{(m+1)!}}\left(\frac{z}{\sqrt{2}l_b}\right)^{m+1}\left|\varphi_{M,m}\right\rangle\left\langle\psi_{z,M}\right| = z\Pi_{z,M}\\
&\Pi_{z,M}\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\Pi_{M,y}\bar{y}dy = \Pi_{z,M}\bar{z}
\end{aligned}$$

in (5.17), we see that the last terms vanish and obtain

$$\mathrm{Tr}\left[\eta_b(z)\left(\mathbf{X}-\mathbf{z}\right)\Pi_{z,M}\left(\overline{\mathbf{X}-\mathbf{z}}\right)\right] = 2l_b^2 M\,\mathrm{Tr}\left[\eta_b(z)\Pi_{z,M-1}\right]$$

With (5.14) we conclude that

$$\mathcal{E}_{r+1,M}(z) = 2l_b^2 M\mathcal{E}_{r,M-1}(z) + 2l_b^2(M+1)\mathcal{E}_{r,M+1}(z).$$

Thus we obtain by induction

$$\mathcal{E}_{r,M}(z) \leq C(r)\left(l_b^2 M\right)^r \sum_{i=-r}^{r} \mathcal{E}_{0,M+i}(z). \tag{5.18}$$

Using that

$$\mathcal{E}_{0,M}(z) = \mathrm{Tr}\left[\gamma_b \Pi_{z,M}\right] \leq M^{-k}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k \Pi_{z,M}\right]$$

from (5.18), we get the desired result

$$\mathcal{E}_{r,M}(z) \leq C(r) l_b^{2r} M^{r-k}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k \Pi_{z,M-r:M+r}\right].$$

□

Now we can provide the

Proof of Proposition 5.1. **Step 1: Estimate of $\mathcal{E}_0^l$.** We introduce another projector:

$$\mathrm{Tr}\left[\gamma_b\left[\Pi_{z,\leq M}, \mathcal{V}_{z,l}\right]\right] = \mathrm{Tr}\left[\gamma_b \Pi_{z,\leq M}\left[\Pi_{z,\leq M}, \mathcal{V}_{z,l}\right]\right] + \mathrm{Tr}\left[\Pi_{z,\leq M}\gamma_b\left[\Pi_{z,\leq M}, \mathcal{V}_{z,l}\right]\right]$$

so with (4.7) applied to $A_z := \left[\Pi_{z,\leq M}, \mathcal{V}_{z,l}\right]$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_0^l(z)dz\right| \\
=&\frac{1}{l_b^2}\left|\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2} \varphi(z)\left(\mathrm{Tr}\left[\gamma_b \Pi_{z,\leq M}\left[\Pi_{z,\leq M}, \mathcal{V}_{z,l}\right]\right] + \mathrm{Tr}\left[\Pi_{z,\leq M}\gamma_b\left[\Pi_{z,\leq M}, \mathcal{V}_{z,l}\right]\right]\right)dz\right| \\
\leq&\frac{\sqrt{2\pi}}{l_b}\sqrt{\frac{\|\varphi\|_{L^1}\|\varphi\|_{L^\infty}}{2\pi l_b^2}\sum_{n=0}^{M}\frac{\left\|\mathrm{Tr}\left[\left[\Pi_{\bullet,\leq M}, \mathcal{V}_\bullet\right]^2 \Pi_{\bullet,n}\right]\right\|_{L^\infty}}{(n+1)^k}}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} \\
=&\frac{\sqrt{\|\varphi\|_{L^1}\|\varphi\|_{L^\infty}}}{l_b^2}\sqrt{\sum_{n=0}^{M}\frac{\left\|\|\Pi_{\bullet,n}, \mathcal{V}_\bullet\|_{\mathfrak{L}^2}^2\right\|_{L^\infty}}{(n+1)^k}}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}
\end{aligned} \tag{5.19}$$

We estimate the Hilbert-Schmidt norm with the changes of variables

$$\frac{x-z}{\sqrt{2}l_b} \to x, \quad \frac{y-z}{\sqrt{2}l_b} \to y.$$

This gives

$$\begin{aligned}
\left\|\left[\Pi_{z,n}, \mathcal{V}_{z,l}\right]\right\|_{\mathfrak{L}^2}^2 =& \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left|\left[\Pi_{z,n}, \mathcal{V}_{z,l}\right](x,y)\right|^2 dxdy = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(\mathcal{V}_{z,l}(x) - \mathcal{V}_{z,l}(y)\right)^2 \left|\Pi_{z,n}(x,y)\right|^2 dxdy \\
=&\frac{1}{(2\pi n! l_b^2)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \left(\mathcal{V}_{z,l}(x) - \mathcal{V}_{z,l}(y)\right)^2 \left|\frac{(x-z)(y-z)}{2l_b^2}\right|^{2n} e^{-\frac{|x-z|^2+|y-z|^2}{2l_b^2}} dxdy \\
=&\frac{1}{(\pi n!)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \left(\mathcal{V}_{z,l}(z+\sqrt{2}l_b x) - \mathcal{V}_{z,l}(z+\sqrt{2}l_b y)\right)^2 |xy|^{2n} e^{-|x|^2-|y|^2} dxdy
\end{aligned}$$

Using the expansion from Equation (5.4),

$$\left|\mathcal{V}_{z,l}(z+\sqrt{2}l_b x)\right| \leq \frac{1}{(l+1)!}\left\|d^{l+1}W\right\|_{L^\infty}\left|\sqrt{2}l_b x\right|^{l+1} \leq e^{\sqrt{2}}\left\|d^{l+1}W\right\|_{L^\infty}\left|l_b x\right|^{l+1}$$

and similarly with y instead of x. With Lemma 5.5,

$$\begin{aligned}\left\|\left[\Pi_{z,n}, \mathcal{V}_{z,l}\right]\right\|_{\mathfrak{L}^2}^2 \leq & C\frac{\left\|d^{l+1}W\right\|_{L^\infty}^2}{n!^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} l_b^{2(l+1)}\left(|x|^{2(l+1)}+|y|^{2(l+1)}\right)|xy|^{2n}e^{-|x|^2-|y|^2}dxdy\\ \leq & C\left\|d^{l+1}W\right\|_{L^\infty}^2\left((n+1)l_b^2\right)^{l+1}\end{aligned}$$

Inserting this in (5.19),

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_0^l(z)dz\right| \leq & C\sqrt{\|\varphi\|_{L^1}\|\varphi\|_{L^\infty}}\left\|d^{l+1}W\right\|_{L^\infty}\frac{1}{l_b^2}\sqrt{\sum_{n=0}^{M}(n+1)^{l+1-k}l_b^{2(l+1)}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\\ = & C\sqrt{\|\varphi\|_{L^1}\|\varphi\|_{L^\infty}}\left\|d^{l+1}W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\,l_b^{l-1}\sqrt{p_{k-(l+2)}(M)} \qquad (5.20)\end{aligned}$$

with $p_{k-(l+2)}(M)$ as in (4.12).

Step 2: Estimate of $\mathcal{E}_{\mathrm{II}}^{p,q}$. From (5.7),

$$\mathcal{E}_{\mathrm{II}}^{p,q}(z) := \frac{\sqrt{M+1}}{2\sqrt{2}\pi l_b^3 p!}\sum_{i_1\dots i_p\in\{1,2\}^p}\partial_{i_1}\dots\partial_{i_p}W(z)i^{i_q-1}$$

$$\mathrm{Tr}\left[\gamma_b\left(\prod_{j=1}^{q-1}(X_{i_j}-z_{i_j})\right)\left((-1)^{i_q-1}\left|\psi_{z,M}\right\rangle\left\langle\psi_{z,M+1}\right|+\left|\psi_{z,M+1}\right\rangle\left\langle\psi_{z,M}\right|\right)\left(\prod_{j=q+1}^{p}(X_{i_j}-z_{i_j})\right)\right]$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{\mathrm{II}}^{p,q}(z)dz\right| \leq & C(p)\left\|d^pW\right\|_{L^\infty}\|\varphi\|_{L^\infty}\frac{\sqrt{M}}{l_b^3}\\ & \int_{\mathbb{R}^2}\left(\epsilon\mathcal{E}_{q-1,M}(z)+\epsilon\mathcal{E}_{q-1,M+1}(z)+\frac{1}{\epsilon}\mathcal{E}_{p-q,M}(z)+\frac{1}{\epsilon}\mathcal{E}_{p-q,M+1}(z)\right)dz\end{aligned} \qquad (5.21)$$

in the notation of Lemma 5.4. Inserting the bound (5.14) leads to

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{\mathrm{II}}^{p,q}(z)dz\right| \leq & C(p)\left\|d^pW\right\|_{L^\infty}\|\varphi\|_{L^\infty}\frac{M^{\frac{1}{2}-k}}{l_b}\left(\epsilon\left(l_b^2M\right)^{q-1}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{M-q+1:M+q}\right]\right.\\ & \left.+\frac{1}{\epsilon}\left(l_b^2M\right)^{p-q}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{M-p+q:M+p-q+1}\right]\right)\end{aligned}$$

Since $1\leq q\leq p$, choosing $\epsilon := \left(l_b^2M\right)^{\frac{p+1}{2}-q}$, we conclude

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{\mathrm{II}}^{p,q}(z)dz\right| \leq C(p)\left\|\nabla W\right\|_{L^\infty}\|\varphi\|_{L^\infty}M^{\frac{p}{2}-k}l_b^{p-2}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{M-p+1:M+p}\right] \qquad (5.22)$$

Step 3: Estimate of $\mathcal{E}_{\mathrm{I}}^{p,q}$. In order to estimate Equation (5.6), we start with an integration by parts for which the following preparations will be helpful.

Let $\odot$ denote the tensor contraction defined for $n,m\geq k$ by

$$u_1\otimes\cdots\otimes u_n\odot^k v_1\otimes\cdots\otimes v_m := \langle u_n|v_1\rangle\dots\left\langle u_{n-k+1}|v_k\right\rangle u_1\otimes\cdots\otimes u_{n-k}\otimes v_{k+1}\otimes\cdots\otimes v_m$$

Identifying $d^pW(z)$ with the associated rank p tensor, we notice that

$$\begin{aligned}&d^pW_z(x-z)^{\otimes(q-1)}\otimes\nabla_z^\perp\Pi_{z,\le M}(x,y)\otimes(y-z)^{\otimes(p-q)}\\=&\nabla^{\otimes p}W(z)\odot^p(x-z)^{\otimes(q-1)}\otimes\nabla_z^\perp\Pi_{z,\le M}(x,y)\otimes(y-z)^{\otimes(p-q)}\\=&\nabla_z^\perp\Pi_{z,\le M}(x,y)\cdot\nabla^{\otimes p}W(z)\odot^{p-1}(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q)}\end{aligned}$$

and

$$\begin{aligned}&\nabla_z^\perp\cdot\nabla^{\otimes p}W(z)\odot^{p-1}(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q)}\\=&\left(\nabla^\perp\cdot\nabla^{\otimes p}\right)W(z)\odot^{p-1}(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q)}\\&+\nabla^{\otimes p}W(z)\odot^p\left(\nabla_z^\perp\otimes(x-z)\right)\otimes(x-z)^{\otimes(q-2)}\otimes(y-z)^{\otimes(p-q)}\\&+\dots\\&+\nabla^{\otimes p}W(z)\odot^p(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q-1)}\otimes\left(\nabla_z^\perp\otimes(y-z)\right)\end{aligned}$$

but because $\nabla^\perp\cdot\nabla=0, \nabla_z^\perp\otimes(x-z)=\nabla_z^\perp\otimes(y-z)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$,

$$\begin{aligned}&\nabla_z^\perp\cdot\nabla^{\otimes p}W(z)\odot^{p-1}(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q)}\\=&\left(\nabla^\perp\cdot\nabla\right)\nabla^{\otimes(p-1)}W(z)\odot^{p-1}(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q)}\\&+\nabla^{\otimes p}W(z)\odot^p\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes(x-z)^{\otimes(q-2)}\otimes(y-z)^{\otimes(p-q)}\\&+\dots\\&+\nabla^{\otimes p}W(z)\odot^p(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q-1)}\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\\=&\left(\nabla^{\otimes p}W(z)\odot^2\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\odot^{p-2}(x-z)^{\otimes(q-2)}\otimes(y-z)^{\otimes(p-q)}\\&+\dots\\&+\left(\nabla^{\otimes p}W(z)\odot^{p-2}(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q-1)}\right)\odot^2\begin{pmatrix}0&1\\-1&0\end{pmatrix}\\=&0.\end{aligned}$$

Indeed, since

$$\nabla^{\otimes p}W(z),\dots,\nabla^{\otimes p}W(z)\odot^{p-2}(x-z)^{\otimes(q-1)}\otimes(y-z)^{\otimes(p-q-1)}$$

are symmetric tensors and $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ is antisymmetric their contraction product is null.

In view of these considerations, an integration by parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{\mathrm{I}}^{p,q}(z)dz \\
=&-\frac{1}{2\pi l_b^2 p!}\int_{\mathbb{R}^2} \mathrm{Tr}\left[\gamma_b d^p W_z \left(X-z\right)^{\otimes(q-1)} \otimes \nabla^\perp \varphi(z)\Pi_{z,\leq M} \otimes (X-z)^{\otimes(p-q)}\right] dz \\
=&-\frac{1}{2\pi l_b^2 p!}\sum_{i_{1:p}\in\{1,2\}^p}\int_{\mathbb{R}^2} \partial_{i_{1:p}} W(z)\nabla^\perp_{i_q}\varphi(z) \\
&\cdot \mathrm{Tr}\left[\gamma_b\left(\prod_{j=1}^{q-1}\left(X_{i_j}-z_{i_j}\right)\right)\Pi_{z,\leq M}\left(\prod_{j=q+1}^{p}\left(X_{i_j}-z_{i_j}\right)\right)\right] dz
\end{aligned} \tag{5.23}$$

Using the Cauchy-Schwarz inequality and (5.13) leads to

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{\mathrm{I}}^{p,q}(z)dz\right| \\
\leq&\frac{C(p)}{2\pi l_b^2}\left\|d^p W\right\|_{L^\infty}\left\|\nabla\varphi\right\|_{L^\infty}\sum_{n=0}^{M}\int_{\mathbb{R}^2}\left(\epsilon\mathcal{E}_{q-1,n}(z)+\frac{1}{\epsilon}\mathcal{E}_{p-q,n}(z)\right)dz \\
\leq&C(p)\left\|d^p W\right\|_{L^\infty}\left\|\nabla\varphi\right\|_{L^\infty}\sum_{n=0}^{M}\left(\epsilon_n l_b^{2(q-1)}n^{q-1-k}+\frac{1}{\epsilon_n}l_b^{2(p-q)}n^{p-q-k}\right)\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{\leq M+p-1}\right]
\end{aligned}$$

Choosing $\epsilon_n := \left(l_b^2 n\right)^{\frac{p+1}{2}-q}$, we conclude

$$\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{\mathrm{I}}^{p,q}(z)dz\right| \leq C(p)\left\|d^p W\right\|_{L^\infty}\left\|\nabla\varphi\right\|_{L^\infty} l_b^{p-1} p_{k-\frac{p+1}{2}}(M)\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{\leq M+p-1}\right] \tag{5.24}$$

Step 4: Conclusion. Putting Equations (5.3), (5.20), (5.22) and (5.24) together we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi(z)\left(\partial_t\rho_{\gamma_b}^{sc,\leq M}(t,z)+\nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc,\leq M}(t,z)\right)dz\right| \\
\leq&\left(C\sqrt{\left\|\varphi\right\|_{L^1}\left\|\varphi\right\|_{L^\infty}}\left\|d^{l+1}W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b^{l-1}\sqrt{p_{k-(l+2)}(M)}\right. \\
&+\sum_{p=1}^{l}\sum_{q=1}^{p}C(p)\left\|\nabla W\right\|_{L^\infty}\left\|\varphi\right\|_{L^\infty}M^{\frac{p}{2}-k}l_b^{p-2}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{M-p+1:M+p}\right] \\
&\left.+\sum_{p=2}^{l}\sum_{q=1}^{p}C(p)\left\|d^p W\right\|_{L^\infty}\left\|\nabla\varphi\right\|_{L^\infty}l_b^{p-1}p_{k-\frac{p+1}{2}}(M)\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{\leq M+p-1}\right]\right) \\
\leq&C(l)\left\|\varphi\right\|_{L^1\cap W^{1,\infty}}\left\|W\right\|_{W^{l+1,\infty}}\left(\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b^{l-1}\sqrt{p_{k-(l+2)}(M)}\right. \\
&\left.+\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{M-l+1:M+l}\right]\sum_{p=1}^{l}M^{\frac{p}{2}-k}l_b^{p-2}+\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\sum_{p=2}^{l}l_b^{p-1}p_{k-\frac{p+1}{2}}(M)\right)
\end{aligned}$$

and then

$$[...] \leq C(l) \|\varphi\|_{L^1 \cap W^{1,\infty}} \|W\|_{W^{l+1,\infty}} \left(\sqrt{\operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b^{l-1} \begin{cases} M^{1+\frac{l-k}{2}} & \text{if } k < l+2 \\ \sqrt{\ln(M)} & \text{if } k = l+2 \\ 1 & \text{if } k > l+2 \end{cases} \right.$$

$$\left. + \frac{\operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k \Pi_{M-l+1:M+l}\right]}{l_b M^{k-\frac{1}{2}}} + \operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\right] \sum_{p=2}^{l} l_b^{p-1} \begin{cases} M^{\frac{p+1}{2}-k} & \text{if } k < \frac{p+1}{2} \\ \ln(M) & \text{if } k = \frac{p+1}{2} \\ 1 & \text{if } k > \frac{p+1}{2} \end{cases} \right)$$

where we used $l_b^2 M \leq 1$ for the last inequality. □

5.2. **Dynamics of the first reduced density.** In this part we conclude the proof of Theorem 2.3. What is left to do is to put together the estimates of Sections 4 and 5.1. We start by explaining how to fix the Landau level cut-off M, as hinted at in Remark 5.2.

Lemma 5.6 (**Fixing the Landau level cut-off**).
Let $\alpha > 0$, $k \geq 0, \varphi \in L^1\left(\mathbb{R}_+\right), \gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ and assume

$$\forall t \in \mathbb{R}_+, \gamma_b(t) \geq 0, \operatorname{Tr}\left[\gamma_b(t)\right] = 1$$
$$\int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k\right] dt < \infty$$

then $\exists M(\varphi) \in \left[\!\left[\left\lfloor l_b^{-\alpha} \right\rfloor, 2\left\lfloor l_b^{-\alpha} \right\rfloor \right]\!\right]$ such that

$$\int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k \Pi_{M(\varphi)-l+1:M(\varphi)+l}\right] dt \leq C(l) l_b^\alpha \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k\right] dt$$

Proof. Assume for contradiction that $\forall M \in \left[\!\left[\left\lfloor l_b^{-\alpha} \right\rfloor, 2\left\lfloor l_b^{-\alpha} \right\rfloor \right]\!\right]$,

$$\int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k \Pi_{M-l+1:M+l}\right] dt > \frac{4l}{M} \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k\right] dt$$

then

$$\begin{aligned}
\int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k\right] dt \geq& \frac{1}{2l} \sum_{M=\left\lfloor l_b^{-\alpha} \right\rfloor}^{2\left\lfloor l_b^{-\alpha} \right\rfloor} \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k \Pi_{M-l+1:M+l}\right] dt \\
>& \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k\right] dt \sum_{M=\left\lfloor l_b^{-\alpha} \right\rfloor}^{2\left\lfloor l_b^{-\alpha} \right\rfloor} \frac{2}{M} \\
\geq& \frac{\left\lfloor l_b^{-\alpha} \right\rfloor + 1}{\left\lfloor l_b^{-\alpha} \right\rfloor} \int_{\mathbb{R}_+} |\varphi(t)| \operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k\right] dt
\end{aligned}$$

which yields the desired contradiction. □

Let us now summarize some findings of the previous subsection:

Lemma 5.7 (**Summary of semi-classical approximation**).
Let $\frac{2}{3} < \alpha < 1$ *and* $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ *be a solution of* (2.9) *and assume*

$$\begin{aligned}
&\operatorname{Tr}\left[\gamma_b(0)\right] = 1, 0 \leq \gamma_b(0) \leq 2\pi l_b^2 \\
&\operatorname{Tr}\left[\gamma_b(0) H_b(0)\right] < \infty \\
&V, w \in W^{4,\infty}\left(\mathbb{R}^2\right)
\end{aligned}$$

Then $\forall \varphi \in L^1\left(\mathbb{R}_+, L^1 \cap W^{1,\infty}\left(\mathbb{R}^2\right)\right), \exists M = \mathcal{O}\left(l_b^{-\alpha}\right)$ *such that*

$$\begin{aligned}
\left|\int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathrm{DRIFT}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc,\leq M}\right)\right| &\leq C \|\varphi\|_{L^1(\mathbb{R}_+, L^1\cap W^{1,\infty}(\mathbb{R}^2))}\left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}}\right) \\
&\left(\left|\operatorname{Tr}\left[\gamma_b(0) H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha - 1}\right) \qquad (5.25)
\end{aligned}$$

Moreover, for $\forall t \geq 0$ *and* $\forall \mu \in L^\infty \cap H^1\left(\mathbb{R}^2\right)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \mu\left(\rho_{\gamma_b}(t) - \rho_{\gamma_b}^{sc,\leq M}(t)\right)\right| \\
\leq &C\left(\|\mu\|_{L^\infty} + \|\nabla\mu\|_{L^2}\right)\left(\left|\operatorname{Tr}\left[\gamma_b(0) H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{\frac{\alpha}{2}} + l_b^{1-\frac{\alpha}{2}}\right) \qquad (5.26)
\end{aligned}$$

Finally, if $p > 3$ *and* $\operatorname{Tr}\left[\gamma_b(t)\,|X|^p\right] < \infty$,

$$W_1\left(\rho_{\gamma_b}(t), \rho_{\gamma_b}^{sc,\leq M}(t)\right) \leq C_1(t, p, V, w)\left(l_b^{1-\frac{\alpha}{2}-\frac{6+\alpha}{2p-4}} + l_b^{\frac{\alpha}{2}}\right) \qquad (5.27)$$

with

$$C_1(t, p, V, w) := C(p)\left(1 + \operatorname{Tr}\left[\gamma_b(t)\,|X|^p\right]\right)\left(\left|\operatorname{Tr}\left[\gamma_b(0) H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)$$

Proof. By Lemma 3.8,

$$\operatorname{Tr}\left[\gamma_b(t)\right] = 1 \text{ and } 0 \leq \gamma_b(t) \leq 2\pi l_b^2$$

then with Lemma 3.9 applied to $W =: V + \frac{1}{2} w \star \rho_{\gamma_b(t)}$ and Lemma 3.9,

$$\begin{aligned}
\operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b\right] &\leq \left|\operatorname{Tr}\left[\gamma_b(t) H_b(t)\right]\right| + \left\|V + \frac{1}{2} w \star \rho_{\gamma_b(t)}\right\|_{L^\infty} \\
&= \left|\operatorname{Tr}\left[\gamma_b(0) H_b(0)\right]\right| + \left\|V + \frac{1}{2} w \star \rho_{\gamma_b(t)}\right\|_{L^\infty} \\
&\leq \left|\operatorname{Tr}\left[\gamma_b(0) H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \qquad (5.28)
\end{aligned}$$

Moreover

$$\left\|V + w \star \rho_{\gamma_b}(t)\right\|_{W^{4,\infty}} \leq \|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}} \qquad (5.29)$$

For this proof we choose $M := M\left(t \mapsto \|\varphi(t)\|_{L^1\cap W^{1,\infty}(\mathbb{R}^2)}\right)$ according to Lemma 5.6.

Using (4.4) for $k = 1$ along with (5.28), $\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b\right]} \le \mathrm{Tr}\left[\gamma_b \mathscr{L}_b\right]$ and $M = \mathcal{O}\left(l_b^{-\alpha}\right)$ gives

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} \mu \left(\rho_{\gamma_b}(t) - \rho_{\gamma_b}^{sc,\le M}(t)\right)\right|\\ \le & C\left(\|\mu\|_{L^\infty} + \|\nabla\mu\|_{L^2}\right)\left(M^{-\frac{1}{2}} + l_b\sqrt{M}\right)\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]\\ \le & C\left(\|\mu\|_{L^\infty} + \|\nabla\mu\|_{L^2}\right)\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{\frac{\alpha}{2}} + l_b^{1-\frac{\alpha}{2}}\right)\end{aligned}$$

which is (5.26).

Integrating Equation (5.1) in time with $W := V + w \star \rho_{\gamma_b}$ and using (5.29),

$$\begin{aligned}&\left|\int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathrm{DRIFT}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc,\le M}\right)\right| \le C\left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}}\right)\\ &\int_{\mathbb{R}_+} \|\varphi(t)\|_{L^1\cap W^{1,\infty}}\left(\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]\left(l_b^2 M^2 + l_b\sqrt{M}\right) + \frac{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{M-2:M+3}\right]}{l_b\sqrt{M}}\right)dt\end{aligned}$$

Then, using Lemma 5.6 and (5.28),

$$\begin{aligned}&\left|\int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathrm{DRIFT}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc,\le M}\right)\right|\\ \le & C\left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}}\right)\left(l_b^{2-2\alpha} + l_b^{1-\frac{\alpha}{2}} + l_b^{\frac{3}{2}\alpha-1}\right)\int_{\mathbb{R}_+}\|\varphi(t)\|_{L^1\cap W^{1,\infty}}\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]dt\\ \le & C\|\varphi\|_{L^1(\mathbb{R}_+,L^1\cap W^{1,\infty}(\mathbb{R}^2))}\left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}}\right)\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\\ &\left(l_b^{2-2\alpha} + l_b^{1-\frac{\alpha}{2}} + l_b^{\frac{3}{2}\alpha-1}\right)\end{aligned}$$

Noticing that

$$\frac{2}{3} < \alpha \implies 2 - 2\alpha \le 1 - \frac{\alpha}{2}$$

we obtain (5.25).

Let $\beta > 0$, we have

$$\frac{2}{3} < \alpha < 1 \implies 4\sqrt{M}l_b = \mathcal{O}\left(l_b^{1-\frac{\alpha}{2}}\right) \ll l_b^{\frac{\alpha}{2}} \ll l_b^{-\beta}$$

so we can apply Proposition (4.4) for $k = 1$ and obtain

$$\begin{aligned}&W_1\left(\rho_{\gamma_b}(t), \rho_{\gamma_b}^{sc,\le M}(t)\right)\\ \le & C(p)\left(1 + \mathrm{Tr}\left[\gamma_b(t)\left|X\right|^p\right]\right)\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]\left(l_b^{1-\beta}\sqrt{M} + M^{-\frac{1}{2}} + l_b^{\beta(p-1)} + Ml_b^{\beta(p-3)-2}\right)\\ \le & C(p)\left(1 + \mathrm{Tr}\left[\gamma_b(t)\left|X\right|^p\right]\right)\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\\ &\left(l_b^{1-\frac{\alpha}{2}-\beta} + l_b^{\frac{\alpha}{2}} + l_b^{\beta(p-1)} + l_b^{\beta(p-3)-2-\alpha}\right)\end{aligned}$$

Remark that

$$\beta(p-1) \ge \beta(p-3) - 2 - \alpha$$

so the optimisation in β is done through

$$\beta(p-3)-2-\alpha=1-\frac{\alpha}{2}-\beta \implies \beta=\frac{3+\frac{\alpha}{2}}{p-2}$$

This choice of β leads to (5.27). □

Next we turn to the

Proof of Theorem 2.3. Let $\varphi \in C_c^\infty\left(\mathbb{R}_+\times\mathbb{R}^2\right)$ and choose M according to Lemma 5.7.

Step 1: decomposition. With an integration by parts,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\varphi(0)\rho_{\gamma_b}(0)-\int_{\mathbb{R}_+\times\mathbb{R}^2}\rho_{\gamma_b}\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)\\
&=\int_{\mathbb{R}^2}\varphi(0)\left(\rho_{\gamma_b}(0)-\rho_{\gamma_b}^{sc,\le M}(0)\right)+\int_{\mathbb{R}^2}\varphi(0)\rho_{\gamma_b}^{sc,\le M}(0)-\int_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}-\rho_{\gamma_b}^{sc,\le M}\right)\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)\\
&\quad-\int_{\mathbb{R}_+\times\mathbb{R}^2}\rho_{\gamma_b}^{sc,\le M}\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)\\
&=\int_{\mathbb{R}^2}\varphi(0)\left(\rho_{\gamma_b}(0)-\rho_{\gamma_b}^{sc,\le M}(0)\right)+\int_{\mathbb{R}_+\times\mathbb{R}^2}\varphi\mathrm{DRIFT}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\le M})\\
&\quad-\int_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}-\rho_{\gamma_b}^{sc,\le M}\right)\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)
\end{aligned}\tag{5.30}$$

Step 2: third term in (5.30)**.** For $t\in\mathbb{R}_+$, (5.26) implies

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\left(\rho_{\gamma_b}(t)-\rho_{\gamma_b}^{sc,\le M}(t)\right)\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t)\right|\le C\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right|+\|V\|_{L^\infty}+\|w\|_{L^\infty}\right)\\
&\left(\left\|\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t)\right\|_{L^\infty}+\left\|\nabla\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t)\right\|_{L^2}\right)\left(l_b^{\frac{\alpha}{2}}+l_b^{1-\frac{\alpha}{2}}\right)
\end{aligned}\tag{5.31}$$

Hence we estimate

$$\begin{aligned}
\left\|\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t)\right\|_{L^\infty}&=\left\|\partial_t\varphi(t)+\nabla^\perp(V+w\star\rho_{\gamma_b(t)})\cdot\nabla\varphi(t)\right\|_{L^\infty}\\
&\le\left\|\partial_t\varphi(t)\right\|_{L^\infty}+\left(\|\nabla V\|_{L^\infty}+\|\nabla w\|_{L^\infty}\right)\|\nabla\varphi(t)\|_{L^\infty}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\nabla\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t)\right\|_{L^2}\\
&=\left\|\partial_t\nabla\varphi(t)+\nabla\left(\nabla^\perp(V+w\star\rho_{\gamma_b(t)})\cdot\nabla\varphi(t)\right)\right\|_{L^2}\\
&\le\left\|\partial_t\nabla\varphi(t)\right\|_{L^2}+\left(\|V\|_{W^{2,\infty}}+\|W\|_{W^{2,\infty}}\right)\|\nabla\varphi(t)\|_{L^2}+\left(\|dV\|_{L^\infty}+\|dw\|_{L^\infty}\right)\left\|d^2\varphi(t)\right\|_{L^2}
\end{aligned}$$

so

$$\begin{aligned}
&\int_{\mathbb{R}_+} \left\|\mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t)\right\|_{L^\infty} + \left\|\nabla \mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)(t)\right\|_{L^2} dt \\
&\leq \left(1 + \|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right) \\
&\quad \int_{\mathbb{R}_+} \left(\left\|\partial_t \varphi(t)\right\|_{L^\infty} + \|\nabla\varphi(t)\|_{L^\infty} + \left\|\partial_t \nabla\varphi(t)\right\|_{L^2} + \|\nabla\varphi(t)\|_{L^2} + \left\|d^2\varphi(t)\right\|_{L^2}\right) dt \\
&\leq \left(1 + \|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right) \|\varphi\|_{W^{1,1}(\mathbb{R}_+, W^{1,\infty}\cap H^2(\mathbb{R}^2))}
\end{aligned}$$

thus (5.31) implies

$$\begin{aligned}
&\left|\int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leq M}\right) \mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)\right| \leq C\, \|\varphi\|_{W^{1,1}(\mathbb{R}_+, W^{1,\infty}\cap H^2(\mathbb{R}^2))} \left(1 + \|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right) \\
&\left(\left|\mathrm{Tr}\left[\gamma_b(0) H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right) \left(l_b^{\frac{\alpha}{2}} + l_b^{1-\frac{\alpha}{2}}\right) \qquad (5.32)
\end{aligned}$$

Step 3: conclusion.

Inserting (5.26) for $\mu := \varphi(0)$, (5.25) and (5.32) in (5.30), we get

$$\left|\int_{\mathbb{R}^2} \varphi(0)\rho_{\gamma_b}(0) - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b} \mathrm{DRIFT}_{\rho_{\gamma_b}}(\varphi)\right| \leq C(\varphi, V, w) \left(l_b^{\frac{\alpha}{2}} + l_b^{1-\frac{\alpha}{2}} + l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha - 1}\right)$$

Finally,

$$\alpha \mapsto \min\left(\frac{\alpha}{2}, 1 - \frac{\alpha}{2}, 2 - 2\alpha, \frac{3}{2}\alpha - 1\right)$$

is maximal at $2 - 2\alpha = -1 + \frac{3}{2}\alpha$, so we conclude by taking

$$\alpha := \frac{6}{7}.$$

□

6. Stability for perturbed classical flows

Our limit model, the drift equation (2.5), enjoys stability estimates with respect to the initial data, by transport arguments à la Dobrushin [17] (see e.g. [23, Section 1.4] for exposition of this material). In this section we include a small source term in this formalism, with the goal of treating the error obtained in Theorem 2.3 in this way. This will provide estimates on the difference between the density of the solution to the quantum evolution (2.2) and the solution of the limit equation, leading to the proof of Theorem 2.4.

6.1. A Dobrushin-type Estimate.

We aim at comparing a solution to the drift equation (2.5) to a solution to a similar equation with a small source term and a possibly different initial datum.

Let S_b be function of time and space. Let ρ_b be the solution to

$$\begin{aligned}
&\partial_t \rho_b + \nabla^\perp V \cdot \nabla\rho_b + \nabla^\perp w \star \rho_b \cdot \nabla\rho_b = S_b(t,z) \\
&\rho_b(t_0, \bullet) =: \rho_{b,0} \in L^1\left(\mathbb{R}^2, \mathbb{R}_+\right), \left\|\rho_{b,0}\right\|_{L^1} = 1 \qquad (6.1)
\end{aligned}$$

and ρ the solution to

$$\partial_t \rho + \nabla^\perp V \cdot \nabla \rho + \nabla^\perp w \star \rho \cdot \nabla \rho = 0$$
$$\rho(t_0, \bullet) =: \rho_0 \in L^1\left(\mathbb{R}^2, \mathbb{R}_+\right), \|\rho_0\|_{L^1} = 1 \tag{6.2}$$

Let $Z_\rho(t, t_0, z_0)$ be the flow defined by

$$\partial_t Z_\rho(t, t_0, z_0) = \nabla^\perp V\left(Z_\rho(t, t_0, z_0)\right) + \nabla^\perp w \star \rho(t, \bullet)\left(Z_\rho(t, t_0, z_0)\right) \tag{6.3}$$
$$\partial_t Z_\rho(t_0, t_0, z_0) = z_0 \tag{6.4}$$

We claim the following, which is classical for $S_b(t, z) \equiv 0$

Proposition 6.1 (**A Dobrushin-type Estimate**).
With the notation above,

$$W_1(\rho_b(t), \rho(t)) \leq e^{2\left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right)|t - t_0|}\left(W_1(\rho_{b,0}, \rho_0) + \mathcal{E}_{S_b}(t)\right)$$

with

$$\mathcal{E}_{S_b}(t) := \left|\int_{t_0}^{t}\int_{t_0}^{t}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \nabla^\perp w\left(Z_{\rho_b}(\tau, t_0, y) - Z_{\rho_b}(\tau, \nu, x)\right) S_b(\nu, x)\, \rho_{b,0}(y) d\tau d\nu dx dy\right| \tag{6.5}$$

We will use characteristics for the above equations. For a general, time-dependent, potential W the PDE

$$\partial_t \rho(t, z) + \nabla^\perp W(t, z) \cdot \nabla_z \rho(t, z) = 0$$

is a transport equation with velocity field $\nabla^\perp W$, thus we define the flow

$$Z : \begin{array}{ccc} \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 & \to & \mathbb{R}^2 \\ t, t_0, z_0 & \mapsto & Z(t, t_0, z_0) \end{array}$$

as the unique (by the Cauchy-Lipschitz theorem) solution of the ODEs

$$\partial_t Z(t, t_0, z_0) = \nabla^\perp W(t, Z(t, t_0, z_0))$$
$$Z(t_0, t_0, z_0) = z_0 \tag{6.6}$$

We denote $Z(t, t_0, \bullet)_* \sigma$ the push-forward of a measure σ by the above. We then have the classic

Lemma 6.2 (**Characteristics**).
Let

$$\rho(t, \bullet) := Z(t, t_0, \bullet)_* \rho_0 + \int_{t_0}^{t} Z(t, \tau, \bullet)_* S(\tau, \bullet) d\tau \tag{6.7}$$

then

$$\partial_t \rho(t, z) + \nabla^\perp W(t, z) \cdot \nabla_z \rho(t, z) = S(t, z)$$
$$\rho(t_0, \bullet) = \rho_0$$

Proof. Let $t, \tau \in \mathbb{R}$ the flow satisfies

$$Z(t, \tau, Z(\tau, t, \bullet)) = \mathrm{Id}_{\mathbb{R}^2} \tag{6.8}$$

so

$$Z(t, \tau, \bullet)^{-1} = Z(\tau, t, \bullet)$$

Thus (6.7) can we rewritten

$$\rho(t,z) = \rho_0(Z(t_0,t,z)) + \int_{t_0}^t S(\tau, Z(\tau,t,z))d\tau$$

Now compute

$$\begin{aligned}\partial_t\rho(t,z) =&\partial_{t_0}Z(t_0,t,z)\cdot\nabla\rho_0\left(Z(t_0,t,z)\right) + S\left(t, Z(t,t,z)\right)\\ &+\int_{t_0}^t \partial_{t_0}Z(\tau,t,z)\cdot\nabla_z S\left(\tau, Z(\tau,t,z)\right)d\tau\\ =&S(t,z)+\partial_{t_0}Z(t_0,t,z)\cdot\nabla\rho_0\left(Z(t_0,t,z)\right)+\int_{t_0}^t \partial_{t_0}Z(\tau,t,z)\cdot\nabla_z S\left(\tau,Z(\tau,t,z)\right)d\tau\end{aligned}$$

and

$$\begin{aligned}\nabla^\perp W(t,z)\cdot\nabla_z\rho(t,z) =&d\rho(t,\bullet)_z\nabla^\perp W(t,z)\\ =&d\rho_{0\,Z(t_0,t,z)}dZ(t_0,t,\bullet)_z\nabla^\perp W(t,z)\\ &+\int_{t_0}^t dS(\tau,\bullet)_{Z(\tau,t,z)}dZ(\tau,t,\bullet)_z\nabla^\perp W(t,z)d\tau\\ =&\nabla\rho_0(Z(t_0,t,z))\cdot dZ(t_0,t,\bullet)_z\nabla^\perp W(t,z)\\ &+\int_{t_0}^t \nabla_z S(\tau,Z(\tau,t,z))\cdot dZ(\tau,t,\bullet)_z\nabla^\perp W(t,z)d\tau\end{aligned}$$

Together the above give

$$\begin{aligned}&\partial_t\rho(t,z)+\nabla^\perp W(t,z)\cdot\nabla_z\rho(t,z) = S(t,z)\\ &+\nabla\rho_0(Z(t_0,t,z))\cdot\Big(\partial_{t_0}Z(t_0,t,z)+dZ(t_0,t,\bullet)_z\nabla^\perp W(t,z)\Big)\\ &+\int_{t_0}^t\nabla_z S(\tau,Z(\tau,t,z))\cdot\Big(\partial_{t_0}Z(\tau,t,z)+dZ(\tau,t,\bullet)_z\nabla^\perp W(t,z)\Big)d\tau \qquad (6.9)\end{aligned}$$

But (6.8) also implies

$$\begin{aligned}&dZ(t,\tau,\bullet)_{Z(\tau,t,z)}dZ(\tau,t,\bullet)_z = \mathrm{Id}_{\mathbb{R}^2}\\ &\frac{d}{dt}Z(t,\tau,Z(\tau,t,z)) = \partial_t Z(t,\tau,Z(\tau,t,z)) + dZ(t,\tau,\bullet)_{Z(\tau,t,z)}\partial_{t_0}Z(\tau,t,z) = 0\end{aligned}$$

therefore, combining with (6.6) and (6.8) leads to

$$\begin{aligned}\partial_{t_0}Z(\tau,t,z) =& -\left(dZ(t,\tau,\bullet)_{Z(\tau,t,z)}\right)^{-1}\partial_t Z(t,\tau,Z(\tau,t,z)) = -dZ(\tau,t,\bullet)_z\partial_t Z(t,\tau,Z(\tau,t,z))\\ =& -dZ(\tau,t,\bullet)_z\nabla^\perp W\left(t, Z\left(t,\tau,Z(\tau,t,z)\right)\right) = -dZ(\tau,t,\bullet)_z\nabla^\perp W(t,z)\end{aligned}$$

so we conclude by seeing that the last two terms in (6.9) are null. □

Next we introduce couplings as in (2.11) to state the

Lemma 6.3 (**Coupling**).
Let ρ_b and ρ be as in (6.1) *and* (6.2). *Let π_b we a coupling between $\rho_{b,0}$ and ρ_0 and introduce the notation*

$$D_{\pi_b}(\tau) := \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2}\left|Z_{\rho_b}(\tau,t_0,x) - Z_\rho(\tau,t_0,y)\right|d\pi_b(x,y)$$

then

$$D_{\pi_b}(t) \leq \left(D_{\pi_b}(t_0) + \mathcal{E}_{S_b}(t) \right) e^{2(\|V\|_{W^{2,\infty}} + \|W\|_{W^{2,\infty}})|t-t_0|}$$

with the same notation as in (6.5)*.*

Proof. Starting from (6.3), (6.4),

$$\begin{aligned}
&Z_{\rho_b}(t,t_0,z_{b,0}) - Z_\rho(t,t_0,z_0) - \left(z_{b,0} - z_0\right) \\
&= \int_{t_0}^t \nabla^\perp \left(V + w \star \rho_b(\tau,\bullet)\right) \left(Z_{\rho_b}(\tau,t_0,z_{b,0})\right) d\tau - \int_{t_0}^t \nabla^\perp \left(V + w \star \rho(\tau,\bullet)\right) \left(Z_\rho(\tau,t_0,z_0)\right) d\tau \\
&= \int_{t_0}^t \left(\nabla^\perp V \left(Z_{\rho_b}(\tau,t_0,z_{b,0})\right) - \nabla^\perp V \left(Z_\rho(\tau,t_0,z_0)\right)\right) d\tau \\
&+ \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - x\right) \rho_b(\tau,x) d\tau dx - \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_\rho(\tau,t_0,z_0) - y\right) \rho(\tau,y) d\tau dy
\end{aligned}$$

Using Lemma 6.2,

$$\rho_b(\tau,x) = \rho_{b,0}\left(Z_{\rho_b}(t_0,\tau,x)\right) + \int_{t_0}^t S_b\left(\nu, Z_{\rho_b}(\nu,\tau,x\right) d\nu, \quad \rho(\tau,y) = \rho_0\left(Z_\rho(t_0,\tau,y)\right)$$

and inserting this in the above leads to

$$\begin{aligned}
&Z_{\rho_b}(t,t_0,z_{b,0}) - Z_\rho(t,t_0,z_0) - (z_{b,0} - z_0) \\
&= \int_{t_0}^t \left(\nabla^\perp V \left(Z_{\rho_b}(\tau,t_0,z_{b,0})\right) - \nabla^\perp V \left(Z_\rho(\tau,t_0,z_0)\right)\right) d\tau \\
&+ \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - x\right) \rho_{b,0}\left(Z_{\rho_b}(t_0,\tau,x)\right) d\tau dx \\
&- \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_\rho(\tau,t_0,z_0) - y\right) \rho_0\left(Z_\rho(t_0,\tau,y)\right) d\tau dy \\
&+ \int_{t_0}^t \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w \left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - x\right) S_b\left(\nu, Z_{\rho_b}(\nu,\tau,x)\right) d\tau d\nu dx
\end{aligned}$$

Since the flow is divergence free it preserves volume, thus with the changes of variable

$$Z_{\rho_b}(t_0,\tau,x) \mapsto x, Z_\rho(t_0,\tau,y) \mapsto y$$

and again $Z_{\rho_b}(\nu,\tau,x) \mapsto x$ for the last integral,

$$
\begin{aligned}
& Z_{\rho_b}(t,t_0,z_{b,0}) - Z_\rho(t,t_0,z_0) - (z_{b,0}-z_0) \\
=& \int_{t_0}^t \left(\nabla^\perp V\left(Z_{\rho_b}(\tau,t_0,z_{b,0})\right) - \nabla^\perp V\left(Z_\rho(\tau,t_0,z_0)\right)\right) d\tau \\
&+ \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_{\rho_b}(\tau,t_0,x)\right) \rho_{b,0}(x) d\tau dx \\
&- \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w\left(Z_\rho(\tau,t_0,z_0) - Z_\rho(\tau,t_0,y)\right) \rho_0(y) d\tau dy \\
&+ \int_{t_0}^t \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_{\rho_b}(\tau,\nu,x)\right) S_b(\nu,x)\, d\tau d\nu dx \\
&= \int_{t_0}^t \left(\nabla^\perp V\left(Z_{\rho_b}(\tau,t_0,z_{b,0})\right) - \nabla^\perp V\left(Z_\rho(\tau,t_0,z_0)\right)\right) d\tau \\
&- \int_{t_0}^t \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left(\nabla^\perp w\left(Z_\rho(\tau,t_0,z_0) - Z_\rho(\tau,t_0,y)\right) - \nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_{\rho_b}(\tau,t_0,x)\right)\right) \\
& d\tau d\pi_b(x,y) + \int_{t_0}^t \int_{t_0}^t \int_{\mathbb{R}^2} \nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_{\rho_b}(\tau,\nu,x)\right) S_b(\nu,x)\, d\tau d\nu dx
\end{aligned}
$$

Next, using

$$
\begin{aligned}
& \left|\nabla^\perp w\left(Z_\rho(\tau,t_0,z_0) - Z_\rho(\tau,t_0,y)\right) - \nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_{\rho_b}(\tau,t_0,x)\right)\right| \\
\le & \left|\nabla^\perp w\left(Z_\rho(\tau,t_0,z_0) - Z_\rho(\tau,t_0,y)\right) - \nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_\rho(\tau,t_0,y)\right)\right| \\
& + \left|\nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_\rho(\tau,t_0,y)\right) - \nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_{\rho_b}(\tau,t_0,x)\right)\right| \\
\le & \|w\|_{W^{2,\infty}} \left(\left|Z_\rho(\tau,t_0,z_0) - Z_{\rho_b}(\tau,t_0,z_{b,0})\right| + \left|Z_\rho(\tau,t_0,y) - Z_{\rho_b}(\tau,t_0,x)\right|\right)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
& \left|Z_{\rho_b}(t,t_0,z_{b,0}) - Z_\rho(t,t_0,z_0)\right| \le \left|z_{b,0}-z_0\right| + \|w\|_{W^{2,\infty}} \int_{t_0}^t \left|Z_\rho(\tau,t_0,z_0) - Z_{\rho_b}(\tau,t_0,z_{b,0})\right| d\tau \\
& + \|w\|_{W^{2,\infty}} \int_{t_0}^t \left|Z_\rho(\tau,t_0,z_0) - Z_{\rho_b}(\tau,t_0,z_{b,0})\right| d\tau \\
& + \|w\|_{W^{2,\infty}} \int_{t_0}^t \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left|Z_\rho(\tau,t_0,y) - Z_{\rho_b}(\tau,t_0,x)\right| d\pi_b(x,y) d\tau \\
& + \int_{t_0}^t \int_{t_0}^t \int_{\mathbb{R}^2} \left|\nabla^\perp w\left(Z_{\rho_b}(\tau,t_0,z_{b,0}) - Z_{\rho_b}(\tau,\nu,x)\right) S_b(\nu,x)\right| d\tau d\nu dx
\end{aligned}
$$

and integrating against $\pi_b(z_{b,0}, z_0)$ we obtain

$$\begin{aligned}
D_{\pi_b}(t) \leq & D_{\pi_b}(0) + \left(\|V\|_{W^{2,\infty}} + 2\|w\|_{W^{2,\infty}}\right) \int_{t_0}^{t} D_{\pi_b}(\tau) d\tau \\
& + \left| \int_{t_0}^{t} \int_{t_0}^{t} \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \nabla^{\perp} w \left(Z_{\rho_b}(\tau, t_0, z_{b,0}) - Z_{\rho_b}(\tau, \nu, x) \right) S_b(\nu, x) \, \rho_{b,0}(z_{b,0}) d\tau d\nu dx dz_{b,0} \right| \\
\leq & D_{\pi_b}(0) + 2\left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right) \int_{t_0}^{t} D_{\pi_b}(\tau) d\tau \\
& + \left| \int_{t_0}^{t} \int_{t_0}^{t} \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \nabla^{\perp} w \left(Z_{\rho_b}(\tau, t_0, y) - Z_{\rho_b}(\tau, \nu, x) \right) S_b(\nu, x) \, \rho_{b,0}(y) d\tau d\nu dx dy \right|
\end{aligned}$$

□

We conclude this subsection by giving the

Proof of Proposition 6.1. Define

$$\begin{aligned}
\phi_t(x, y) &:= \left(Z_{\rho_b}(t, t_0, x), Z_{\rho}(t, t_0, y) \right) \\
\pi_b(t) &:= \phi_{t*} \pi_b
\end{aligned}$$

Then

$$\begin{aligned}
\pi_b(\mathbb{R}^2, A) = & \pi_b \left(Z_{\rho_b}(t_0, t, \mathbb{R}^2), Z_{\rho}(t_0, t, A) \right) = \pi_b(\mathbb{R}^2, Z_{\rho}(t_0, t, A)) = \rho_0(Z_{\rho}(t_0, t, A)) \\
= & Z_{\rho}(t, t_0, \bullet)_* \rho_0(A) = \rho(t, A)
\end{aligned}$$

and similarly for the second variable thus $\pi_b(t)$ is a coupling for $\rho_b(t)$ and $\rho(t)$ and

$$\begin{aligned}
W_1(\rho_b(t), \rho(t)) = & \inf_{\pi \in \Gamma(\rho_b(t), \rho(t))} \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} |x - y| \, d\pi(x, y) \\
= & \inf_{\pi_b \in \Gamma(\rho_{b,0}, \rho_0)} \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} |x - y| \, d\phi_{t*}\pi_b(x, y) \\
= & \inf_{\pi_b \in \Gamma(\rho_{b,0}, \rho_0)} \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left| Z_{\rho_b}(t, t_0, x) - Z_{\rho}(t, t_0, y) \right| d\pi_b(x, y) \\
= & \inf_{\pi_b \in \Gamma(\rho_{b,0}, \rho_0)} D_{\pi_b}(t) \leq e^{2\left(\|w\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right)|t - t_0|} \left(\inf_{\pi_b \in \Gamma(\rho_{b,0}, \rho_0)} D_{\pi_b}(0) + \mathcal{E}_{S_b}(t) \right) \\
= & e^{2\left(\|w\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right)|t - t_0|} \left(W_1(\rho_{b,0}, \rho_0) + \mathcal{E}_{S_b}(t) \right)
\end{aligned}$$

□

6.2. **Application: proof of Theorem 2.4.** As in Section 5 it is convenient to first consider the dynamics of the truncated semi-classical density:

Proposition 6.4 (**Convergence of the semi-classical density in Wasserstein metric**). *Let* $\frac{2}{3} < \alpha < 1$. *Under the same assumptions as in Theorem 2.3, with in addition*

$$\nabla w \in L^1\left(\mathbb{R}^2\right), w \in H^2\left(\mathbb{R}^2\right),$$

if $\rho \in L^\infty\left(\mathbb{R}_+, L^1\left(\mathbb{R}^2\right)\right)$ *solves the drift equation* (2.5) *then* $\exists M = \mathcal{O}\left(l_b^{-\alpha}\right)$ *such that*

$$W_1\left(\rho_{\gamma_b}^{sc,\le M}(t), \rho(t)\right) \le e^{2\left(\|w\|_{W^{2,\infty}}+\|V\|_{W^{2,\infty}}\right)t} \left(W_1\left(\rho_{\gamma_b}^{sc,\le M}(0), \rho(0)\right) + C_2(t,V,w)\left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1}\right)\right)$$

with

$$C_2(t,V,w) := Ct^2\left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}} e^{\left(\|V\|_{W^{2,\infty}}+\|w\|_{W^{2,\infty}}\right)t}\right)\left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}} + \|w\|_{H^2}\right) \left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)$$

Proof. We will apply Proposition 6.1 with

$$S_b := \mathrm{DRIFT}_{\rho_{\gamma_b}^{sc,\le M}}\left(\rho_{\gamma_b}^{sc,\le M}\right)$$

where the drift operator is defined in (2.5). We need to estimate the error term $\mathcal{E}_{S_b}(t)$ coming from (6.5). We define

$$\varphi(t,z) := \int_0^T \int_{\mathbb{R}^2} \nabla^\perp w\left(Z_{\rho_{\gamma_b}}(\tau,0,x) - Z_{\rho_{\gamma_b}}(\tau,t,z)\right)\rho_{\gamma_b}^{sc,\le M}(0,x)d\tau dx$$

so that

$$\begin{aligned}
\mathcal{E}_{S_b}(T) &= \left|\int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathbb{1}_{[0,T]} \mathrm{DRIFT}_{\rho_{\gamma_b}^{sc,\le M}}\left(\rho_{\gamma_b}^{sc,\le M}\right)\right| \\
&\le \left|\int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathbb{1}_{[0,T]} \mathrm{DRIFT}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc,\le M}\right)\right| \\
&\quad + \left|\int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathbb{1}_{[0,T]} \nabla^\perp w \star \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\le M}\right)\cdot \nabla\rho_{\gamma_b}^{sc,\le M}\right| \\
&=: \mathcal{E}_{S_b}^1(T) + \mathcal{E}_{S_b}^2(T).
\end{aligned} \tag{6.10}$$

Step 1: we estimate $\varphi \in L^1\left([0,T], L^1\left(\mathbb{R}^2\right)\cap W^{1,\infty}\left(\mathbb{R}^2\right)\right)$**.** Let $t\in[0,T]$. With the changes of variable $Z_{\rho_{\gamma_b}}(\tau,t,z)\mapsto z$,

$$\begin{aligned}
\|\varphi(t)\|_{L^1} &\le \int_0^T \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left|\nabla^\perp w\left(Z_{\rho_{\gamma_b}}(\tau,0,x) - Z_{\rho_{\gamma_b}}(\tau,t,z)\right)\rho_{b,0}(x)\right| d\tau dx dz \\
&= \int_0^T \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left|\nabla^\perp w\left(Z_{\rho_{\gamma_b}}(\tau,0,x) - z\right)\rho_{b,0}(x)\right| d\tau dx dz \le T\|\nabla w\|_{L^1}
\end{aligned} \tag{6.11}$$

by performing the z integral first. Moreover

$$|\varphi(t,z)| \le T\|\nabla w\|_{L^\infty}$$

and

$$\left|\nabla_z\varphi(t,z)\right| \le \|w\|_{W^{2,\infty}} \int_0^T \left|dZ_{\rho_{\gamma_b}}(\tau,t,\bullet)_z\right| d\tau. \tag{6.12}$$

But, using

$$Z_{\rho_{\gamma_b}}(\tau,t,z) = z + \int_t^\tau \nabla^\perp \left(V + w \star \rho_{\gamma_b}\right)\left(Z_{\rho_{\gamma_b}}(s,t,z)\right) ds$$

we get

$$dZ_{\rho_{\gamma_b}}(\tau,t,\bullet)_z = \mathrm{Id}_{\mathbb{R}^2} + \int_t^\tau d\nabla^\perp \left(V + w \star \rho_{\gamma_b}\right)_{Z_{\rho_{\gamma_b}}(s,t,z)} d\left(Z_{\rho_{\gamma_b}}(s,t,\bullet)_z\right) ds$$

so

$$\left|dZ_{\rho_{\gamma_b}}(\tau,t,\bullet)_z\right| \leq \left|\mathrm{Id}_{\mathbb{R}^2}\right| + \left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right) \int_t^\tau \left|d\left(Z_{\rho_{\gamma_b}}(s,t,\bullet)_z\right)\right| ds$$

Hence, applying Grönwall's lemma,

$$\left|dZ_{\rho_{\gamma_b}}(\tau,t,\bullet)_z\right| \leq \sqrt{2} e^{\left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right)|\tau - t|}$$

With (6.12), we conclude that

$$\begin{aligned} \left|\nabla_z \varphi(t,z)\right| \leq & \sqrt{2}\, \|w\|_{W^{2,\infty}} \int_0^T e^{\left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right)|\tau - t|} d\tau \\ \leq & \sqrt{2}\, \|w\|_{W^{2,\infty}}\, T e^{\left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right) T} \end{aligned}$$

Collecting the above estimates we find

$$\|\varphi(t)\|_{L^1 \cap W^{1,\infty}(\mathbb{R}^2)} \leq CT\left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}}\, e^{\left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right)T}\right)$$

and thus

$$\begin{aligned} \left\|\varphi \mathbb{1}_{[0,T]}\right\|_{L^1(\mathbb{R}_+, L^1 \cap W^{1,\infty}(\mathbb{R}^2))} = & \|\varphi\|_{L^1([0,T], L^1 \cap W^{1,\infty}(\mathbb{R}^2))} \\ \leq & CT^2\left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}}\, e^{\left(\|V\|_{W^{2,\infty}} + \|w\|_{W^{2,\infty}}\right)T}\right) \end{aligned} \tag{6.13}$$

Step 2: bound on $\mathcal{E}^2_{S_b}(T)$. We choose M according to Lemma 5.7. With the symmetry of w,

$$\begin{aligned} \mathcal{E}^2_{S_b}(T) = & \left|\int_{\mathbb{R}_+ \times \mathbb{R}^2} \mathbb{1}_{[0,T]} \nabla\varphi \cdot \nabla^\perp w \star \left(\rho_{\gamma_b} - \rho^{sc,\leq M}_{\gamma_b}\right) \rho^{sc,\leq M}_{\gamma_b}\right| \\ = & \left|\int_{\mathbb{R}_+ \times \mathbb{R}^2} \mathbb{1}_{[0,T]} \left(\rho_{\gamma_b} - \rho^{sc,\leq M}_{\gamma_b}\right)\left(\nabla\varphi \rho^{sc,\leq M}_{\gamma_b}\right) \dot{\star}\, \nabla^\perp w\right|. \end{aligned}$$

Then, using (5.26) with

$$\mu(t,x) := \left(\nabla\varphi(t) \rho^{sc,\leq M}_{\gamma_b(t)}\right) \dot{\star}\, \nabla^\perp w(x) := \int \rho^{sc,\leq M}_{\gamma_b(t)}(y) \nabla\varphi(t,y) \cdot \nabla^\perp w(x-y) dy$$

leads to

$$\begin{aligned} \mathcal{E}^2_{S_b}(T) \leq & C \int_0^T \left(\|\mu(t)\|_{L^\infty} + \|\nabla\mu(t)\|_{L^2}\right) dt \\ & \left(\left|\mathrm{Tr}\left[\gamma_b(0) H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{\frac{\alpha}{2}} + l_b^{1 - \frac{\alpha}{2}}\right). \end{aligned}$$

With Young's convolution inequality,

$$\|\mu(t)\|_{L^\infty} \leq \|\nabla\varphi(t)\|_{L^\infty} \|\nabla w\|_{L^\infty}$$
$$\|\nabla\mu(t)\|_{L^2} \leq \left\|\left(\nabla\varphi(t)\rho^{sc,\leq M}_{\gamma_b(t)}\right)\right\|_{L^1} \|w\|_{H^2} \leq \|\nabla\varphi(t)\|_{L^\infty} \|w\|_{H^2}$$

so

$$\mathcal{E}^2_{S_b}(T) \leq C \left\|\varphi\mathbb{1}_{[0,T]}\right\|_{L^1(\mathbb{R}_+,W^{1,\infty}(\mathbb{R}^2))} \left(\|\nabla w\|_{L^\infty} + \|w\|_{H^2}\right)$$
$$\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{\frac{\alpha}{2}} + l_b^{1-\frac{\alpha}{2}}\right). \quad (6.14)$$

Step 3: conclusion. There remains to estimate $\mathcal{E}^1_{S_b}(T)$. With (5.25),

$$\mathcal{E}^1_{S_b}(T) \leq C \left\|\varphi\mathbb{1}_{[0,T]}\right\|_{L^1(\mathbb{R}_+,L^1\cap W^{1,\infty}(\mathbb{R}^2))} \left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}}\right)$$
$$\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1}\right)$$

We combine this with (6.14) and insert the resulting bound in (6.10):

$$\mathcal{E}_{S_b}(T) \leq C \left\|\varphi\mathbb{1}_{[0,T]}\right\|_{L^1(\mathbb{R}_+,L^1\cap W^{1,\infty}(\mathbb{R}^2))} \left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}} + \|w\|_{H^2}\right)$$
$$\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{\frac{\alpha}{2}} + l_b^{1-\frac{\alpha}{2}} + l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1}\right)$$

Note that

$$\frac{2}{3} < \alpha \implies 2 - 2\alpha \leq 1 - \frac{\alpha}{2}$$
$$\alpha < 1 \implies \frac{3}{2}\alpha - 1 \leq \frac{\alpha}{2}. \quad (6.15)$$

hence using (6.13),

$$\mathcal{E}_{S_b}(T) \leq CT^2 \left(\|\nabla w\|_{L^1} + \|w\|_{W^{2,\infty}} e^{\left(\|V\|_{W^{2,\infty}}+\|w\|_{W^{2,\infty}}\right)T}\right)\left(\|V\|_{W^{4,\infty}} + \|w\|_{W^{4,\infty}} + \|w\|_{H^2}\right)$$
$$\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1}\right)$$

We obtain the desired conclusion by using Proposition 6.1. □

Finally we turn to the

Proof of Theorem 2.4. Let $\frac{2}{3} < \alpha < 1$ and M be chosen as in Proposition 6.4. From (5.26),

$$\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b}(t) - \rho^{sc,\leq M}_{\gamma_b}(t)\right)\right|$$
$$\leq C\left(\|\varphi\|_{W^{1,\infty}} + \|\nabla\varphi\|_{L^2}\right)\left(\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty}\right)\left(l_b^{\frac{\alpha}{2}} + l_b^{1-\frac{\alpha}{2}}\right)$$

On the other hand, using Proposition 6.4 and then (5.27),

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b}^{sc,\leq M} - \rho(t)\right)\right| \\
&\leq \|\nabla\varphi\|_{L^\infty} W_1\left(\rho_{\gamma_b}^{sc,\leq M}(t), \rho(t)\right) \\
&\leq \|\nabla\varphi\|_{L^\infty} e^{2\left(\|w\|_{W^{2,\infty}}+\|V\|_{W^{2,\infty}}\right)t}\left(W_1\left(\rho_{\gamma_b}^{sc,\leq M}(0), \rho(0)\right) + C_2(t,V,w)\left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1}\right)\right) \\
&\leq \|\nabla\varphi\|_{L^\infty} e^{2\left(\|w\|_{W^{2,\infty}}+\|V\|_{W^{2,\infty}}\right)t} \\
&\quad\left(W_1\left(\rho_{\gamma_b}(0), \rho(0)\right) + C_1(0,p,V,w)\left(l_b^{1-\frac{\alpha}{2}-\frac{6+\alpha}{2p-4}} + l_b^{\frac{\alpha}{2}}\right) + C_2(t,V,w)\left(l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1}\right)\right) \\
&\leq \left(\|\varphi\|_{W^{1,\infty}} + \|\nabla\varphi\|_{L^2}\right) e^{2\left(\|w\|_{W^{2,\infty}}+\|V\|_{W^{2,\infty}}\right)t}\left(1 + C_1(0,p,V,w) + C_2(t,v,w)\right) \\
&\quad\left(W_1\left(\rho_{\gamma_b}(0), \rho(0)\right) + l_b^{1-\frac{\alpha}{2}-\frac{6+\alpha}{2p-4}} + l_b^{\frac{\alpha}{2}} + l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1}\right)
\end{aligned}$$

Recalling (6.15) and using the triangle inequality,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b(t)} - \rho(t)\right)\right| \\
&\leq \widetilde{C}(p,t,V,w)\left(\|\varphi\|_{W^{1,\infty}} + \|\nabla\varphi\|_{L^2}\right)\left(W_1\left(\rho_{\gamma_b}(0), \rho(0)\right) + l_b^{1-\frac{\alpha}{2}-\frac{6+\alpha}{2p-4}} + l_b^{2-2\alpha} + l_b^{\frac{3}{2}\alpha-1}\right)
\end{aligned}$$

With

$$\begin{aligned}
\widetilde{C}(p,t,V,w) = &\left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \|w\|_{L^\infty} \\
&+ e^{2\left(\|w\|_{W^{2,\infty}}+\|V\|_{W^{2,\infty}}\right)t}\left(1 + C_1(0,p,V,w) + C_2(t,v,w)\right)
\end{aligned}$$

We conclude with with the following optimisation:

$$\alpha \mapsto \min\left(1 - \frac{\alpha}{2} - \frac{6+\alpha}{2p-4}, 2 - 2\alpha, \frac{3}{2}\alpha - 1\right)$$

is maximal at

$$\alpha := \min\left(2\frac{2p-7}{4p-7}, \frac{6}{7}\right)$$

with maximal value

$$\min\left(2\frac{p-7}{4p-7}, \frac{2}{7}\right).$$

□

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